

CORONAE OF RELATIVELY HYPERBOLIC GROUPS AND COARSE COHOMOLOGIES

TOMOHIRO FUKAYA, SHIN-ICHI OGUNI

ABSTRACT. We construct a corona of a relatively hyperbolic group by blowing-up all parabolic points of its Bowditch boundary. We relate the K -homology of the corona with the K -theory of the Roe algebra, via the coarse assembly map. We also establish a dual theory, that is, we relate the K -theory of the corona with the K -theory of the reduced stable Higson corona via the coarse co-assembly map. For that purpose, we formulate generalized coarse cohomology theories. As an application, we give an explicit computation of the K -theory of the Roe-algebra and that of the reduced stable Higson corona of a non-uniform lattice of rank one symmetric space.

1. INTRODUCTION

1.1. The coarse assembly map and its dual. The coarse category is a category whose objects are proper metric spaces and whose morphisms are close classes of coarse maps. Let X be a proper metric space. There are two covariant functors $X \mapsto KX_*(X)$ and $X \mapsto K_*(C^*(X))$ from the coarse category to the category of \mathbb{Z}_2 -graded Abelian groups. Here the \mathbb{Z}_2 -graded Abelian group $KX_*(X)$ is called the *coarse K -homology* of X , and the C^* -algebra $C^*(X)$ is called the *Roe algebra* of X . Roe [27] constructed the following *coarse assembly map*

$$\mu_*: KX_*(X) \rightarrow K_*(C^*(X)),$$

which is a natural transformation from the coarse K -homology to the K -theory of the Roe algebra. For detail, see also [17], [32] and [18].

On the other hand, there are two contravariant functors $X \mapsto KX^*(X)$ and $X \mapsto K_*(\mathfrak{c}^r(X))$. Here the \mathbb{Z}_2 -graded Abelian group $KX^*(X)$ is called the *coarse K -theory* of

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X and the C^* -algebra $\mathfrak{c}^r(X)$ is called the *reduced stable Higson corona* of X . Emerson and Meyer [9] constructed a dual of the coarse assembly map, which is called the *coarse co-assembly map*,

$$\mu^*: K_{*+1}(\mathfrak{c}^r(X)) \rightarrow KX^*(X).$$

In fact, μ^* is a natural transformation from the K -theory of the reduced stable Higson corona to the coarse K -theory with shifting the grading by one. Those assembly maps are closely related to the analytic Novikov conjecture. See [18, Section 12.6] and [10] for details.

In this paper, we study the case of relatively hyperbolic groups with word metrics.

THEOREM 1.1. *Let G be a finitely generated group which is hyperbolic relative to a finite family of infinite subgroups $\mathbb{P} = \{P_1, \dots, P_k\}$. Suppose that each subgroup P_i admits a finite P_i -simplicial complex which is a universal space for proper actions. Then*

- (a) *if for all $i = 1, \dots, k$, the coarse assembly maps $\mu_*: KX_*(P_i) \rightarrow K_*(C^*(P_i))$ are isomorphisms, then so is the coarse assembly map $\mu_*: KX_*(G) \rightarrow K_*(C^*(G))$,*
- (b) *if for all $i = 1, \dots, k$, the coarse co-assembly maps $\mu^*: K_{*+1}(\mathfrak{c}^r(P_i)) \rightarrow KX^*(P_i)$ are isomorphisms, then so is the coarse co-assembly map $\mu^*: K_{*+1}(\mathfrak{c}^r(G)) \rightarrow KX^*(G)$.*

The authors proved the statement (a) in [12]. In this paper, we prove the statement (b).

1.2. Coarse compactification. Let X be a non-compact proper metric space. The *Higson compactification* hX of X is the maximal ideal space of the C^* -algebra of \mathbb{C} -valued, continuous, bounded functions on X of vanishing variation. (See Definition 2.13.) The *Higson corona* of X is $\nu X = hX \setminus X$. A corona of X is a pair (W, ζ) of a compact metrizable space W and a continuous map $\zeta: \nu X \rightarrow W$. When ζ is surjective, we obtain a compactification $X \cup W$. (See Section 2.2.)

Let (W, ζ) be a corona of X . Then there are certain transgression maps

- (1) $T_W: KX_*(X) \rightarrow \tilde{K}_{*-1}(W);$
- (2) $T_W: \tilde{K}^{*-1}(W) \rightarrow KX^*(X);$
- (3) $T_W: \tilde{H}^{*-1}(W) \rightarrow HX^*(X).$

Here $\tilde{H}^*(W)$ is the reduced cohomology of W and $HX^*(X)$ is the coarse cohomology of X . (See [27].) In Section 3.2, we give a construction of the map (1) which appeared in

[17, Appendix]. The map (2) is constructed in Section 4. The map (3) is constructed in [27, Section 5.3].

There exists a homomorphism $b: K_*(C^*(X)) \rightarrow \tilde{K}_{*-1}(W)$ such that $T_W = b \circ \mu_*$. Therefore if the transgression map (1) is injective, then so is the coarse assembly map for X . It is also known that if (2) or (3) is surjective then the coarse assembly map is rationally injective. For details, see [17, Appendix], [27, (6.32)] and [9, Section 6]. The statement that the transgression map (3) is surjective for some corona W is a version of the Weinberger conjecture. In this paper, we consider transgression maps for relatively hyperbolic groups.

Let G be a finitely generated group and \mathcal{S} be a finite generating set. We suppose that G is hyperbolic relative to a finite family of infinite subgroups $\mathbb{P} = \{P_1, \dots, P_k\}$. Groves and Manning [15] defined the augmented space $X(G, \mathbb{P}, \mathcal{S})$ with a properly discontinuous action of G by isometries. They showed that $X(G, \mathbb{P}, \mathcal{S})$ is hyperbolic in the sense of Gromov. We denote by $\partial X(G, \mathbb{P}, \mathcal{S})$ the Gromov boundary of $X(G, \mathbb{P}, \mathcal{S})$, which is called the Bowditch boundary of (G, \mathbb{P}) . (See [7, Definition 1.4].) Let (W_i, ζ_i) be a corona of P_i . We blow up all parabolic points of $\partial X(G, \mathbb{P}, \mathcal{S})$ by using W_1, \dots, W_k and obtain a corona ∂X_∞ of G . We call ∂X_∞ the blown-up corona of $(G, \mathbb{P}, \{W_1, \dots, W_k\})$. See Section 7 for the details of the construction.

THEOREM 1.2. *Let G be an infinite finitely generated group which is hyperbolic relative to a finite family of infinite subgroups $\mathbb{P} = \{P_1, \dots, P_k\}$. Suppose that each subgroup P_i admits a finite P_i -simplicial complex which is a universal space for proper actions. For $i = 1, \dots, k$, let (W_i, ζ_i) be a corona of P_i . Let ∂X_∞ be the blown-up corona of $(G, \mathbb{P}, \{W_1, \dots, W_k\})$.*

- (a) *If $T_{W_i}: KX_*(P_i) \rightarrow \tilde{K}_{*-1}(W_i)$ is an isomorphism for all $i = 1, \dots, k$, then so is $T_{\partial X_\infty}: KX_*(G) \rightarrow \tilde{K}_{*-1}(\partial X_\infty)$.*
- (b) *If $T_{W_i}: \tilde{K}^{*-1}(W_i) \rightarrow KX^*(P_i)$ is an isomorphism for all $i = 1, \dots, k$, then so is $T_{\partial X_\infty}: \tilde{K}^{*-1}(\partial X_\infty) \rightarrow KX^*(G)$.*
- (c) *If $T_{W_i}: \tilde{H}^{*-1}(W_i) \rightarrow HX^*(P_i)$ is an isomorphism for all $i = 1, \dots, k$, then so is $T_{\partial X_\infty}: \tilde{H}^{*-1}(\partial X_\infty) \rightarrow HX^*(G)$.*

COROLLARY 1.3. *Let G be an infinite finitely generated group which is hyperbolic relative to a finite family of infinite subgroups $\mathbb{P} = \{P_1, \dots, P_k\}$. We suppose that \mathbb{P} satisfies all conditions in Theorem 1.1 and Theorem 1.2. Then we have $K_*(C^*(G)) \cong \tilde{K}_{*-1}(\partial X_\infty)$ and $\tilde{K}^*(\partial X_\infty) \cong K_*(\mathfrak{c}^r(G))$.*

As an application, we give an explicit computation of the K -theory of the Roe-algebra and that of the reduced stable Higson corona of a non-uniform lattice of rank one symmetric space. See Corollary 9.1.

The organization of this paper is as follows. In Section 2, we review the coarse structure and introduce a pull-back coarse structure which plays an essential role in the construction of coronae in Section 7. We also review coronae for proper coarse spaces. In Section 3, we formulate generalized coarse cohomology theories. In Section 4.1, we show that the coarse K -theory [9] satisfies axioms introduced in the previous section. In Section 4.2, we review the construction of the coarse co-assembly map. In Section 5, we show that the coarse co-assembly maps are isomorphisms in the case of proper geodesic spaces which are hyperbolic in the sense of Gromov. In Section 6, we review a definition of relatively hyperbolic groups due to Groves and Manning [15] and give a proof of Theorem 1.1. In Section 7, we construct a corona of a relatively hyperbolic group using a pull-back coarse structure. In Section 8, we give a proof of Theorem 1.2. In Section 9, we give an explicit computation for non-uniform lattices of rank one symmetric spaces. In Appendix A, we study coronae of uniform lattices of Carnot groups which we use for a calculation in Section 9. In Appendix B, we give a proof of the Milnor exact sequence for σ - C^* -algebras, which we often use in the present paper.

2. COARSE COMPACTIFICATION

2.1. Coarse structure. Here we review the coarse structure from [28] and introduce the pullback coarse structure.

Let X be a set. For $E \subset X \times X$, put $E^{-1} := \{(y, x) : (x, y) \in E\}$ and call it the inverse of E . For $E', E'' \subset X \times X$, put $E' \circ E'' := \{(x', x'') : \exists x \in X, (x', x) \in E', (x, x'') \in E''\}$ and call it the product of E' and E'' .

DEFINITION 2.1. A coarse structure on a set X is a collection \mathcal{E} of subsets of $X \times X$, called *controlled sets* for the coarse structure, which contains the diagonal and is closed under the formation of subsets, inverses, products, and finite union. A set equipped with a coarse structure is called a coarse space.

EXAMPLE 2.2. Let X be a metric space. The bounded coarse structure on X is a collection of all subsets $E \subset X \times X$ such that $\sup\{d(x, x') : (x, x') \in E\} < \infty$.

EXAMPLE 2.3. Let G be a countable group. There always exists a proper left invariant metric d on G . The bounded coarse structure on G associated to d does not depend on

the choice of such a metric d . See [28, Proposition 1.15, Example 2.13]. In this paper, we always assume that countable groups are equipped with this canonical coarse structures.

DEFINITION 2.4. Let X be a coarse space and let B be a subset of X . We say that B is *bounded* if $B \times B$ is controlled.

DEFINITION 2.5. Let X be a coarse space and S be a set. Two maps $f, g: S \rightarrow X$ are *close* if the set $\{(f(s), g(s)) : s \in S\} \subset X \times X$ is controlled.

DEFINITION 2.6. Let X and Y be coarse spaces, and let $f: X \rightarrow Y$ be a map.

- (a) The map f is *proper* if the inverse image, under f , of each bounded subset of Y , is also bounded.
- (b) The map f is *bornologous* if for each controlled subset $E \subset X \times X$, the set $f(E)$ is a controlled subset of $Y \times Y$. Here we abbreviate $(f \times f)(E)$ to $f(E)$.
- (c) The map f is *coarse* if it is proper and bornologous.

The spaces X and Y are *coarsely equivalent* if there exist coarse maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f$ and $f \circ g$ are close to the identity maps on X and on Y , respectively. Such a map f is called a coarse equivalence.

DEFINITION 2.7. Let X be a locally compact second countable Hausdorff space. We say that a coarse structure on X is *proper* if

- (a) there is a controlled neighborhood of the diagonal,
- (b) every bounded subset of X is relatively compact, and
- (c) X is coarsely connected, that is, for any pair of points $(x, x') \in X \times X$, the set $\{(x, x')\}$ is controlled.

DEFINITION 2.8. Let X be a set and let Y be a coarse space. Let $f: X \rightarrow Y$ be a map. The *pullback coarse structure* on X is a collection of subsets $E \subset X \times X$ such that $f(E)$ is a controlled subset of $Y \times Y$.

PROPOSITION 2.9. *Let Y be a coarse space. Let X be a set and let $f: X \rightarrow Y$ be a map. We equip X with the pullback coarse structure. Then f is a coarse map. If there exists a map $g: Y \rightarrow X$ such that the composite $f \circ g$ is close to the identity, then X and Y are coarsely equivalent. If Y is coarsely connected, then so is X .*

PROOF. Let \mathcal{E}_Y be a coarse structure of Y . The pullback coarse structure \mathcal{E}_X is the set $\mathcal{E}_X = \{E \subset X \times X : f(E) \in \mathcal{E}_Y\}$. Then it is trivial that f is a coarse map. Suppose

that there exists a map $g: Y \rightarrow X$ such that $f \circ g$ is close to the identity. Then a subset $F = \{(y, f \circ g(y)) : y \in Y\}$ belongs to \mathcal{E}_Y . Let $E \in \mathcal{E}_Y$ be a controlled set. Since $f(g(E)) \subset F^{-1} \circ E \circ F \in \mathcal{E}_Y$, we have $g(E) \in \mathcal{E}_X$. Let $B \subset X$ be a bounded set. Then $g^{-1}(B) \times g^{-1}(B) \subset F \circ f(B \times B) \circ F^{-1} \in \mathcal{E}_Y$, so $g^{-1}(B)$ is bounded. Thus g is a coarse map. Since $f(\{(x, g \circ f(x)) : x \in X\}) \subset F$, we have $g \circ f$ is close to the identity. If Y is coarsely connected, then for any pair of points $(x, x') \in X \times X$, the set $\{(f(x), f(x'))\} \subset Y \times Y$ is controlled, thus so is $\{(x, x')\}$. Therefore X is coarsely connected. \square

DEFINITION 2.10. Let X be a topological space and Y be a metric space. A map $f: X \rightarrow Y$ is *pseudocontinuous* if there exists $\epsilon > 0$ such that for any $x \in X$, the inverse image $f^{-1}(B(f(x); \epsilon))$ of the closed ball of radius ϵ centered at $f(x)$ is a neighborhood of x .

PROPOSITION 2.11. *Let Y be a proper metric space with the bounded coarse structure. Let X be a locally compact second countable Hausdorff space. Let $f: X \rightarrow Y$ be a pseudocontinuous map. We equip X with the pullback coarse structure. If for any compact set $K \subset Y$ the inverse image $f^{-1}(K) \subset X$ is relatively compact, then X is a proper coarse space.*

PROOF. Fix $\epsilon > 0$ satisfying the condition in Definition 2.10. Set $\Delta_\epsilon = \{(x, y) : d(x, y) \leq \epsilon\} \subset Y \times Y$. Then the pullback $f^{-1}(\Delta_\epsilon)$ is a controlled neighborhood of the diagonal. Let $B \subset X$ be a bounded subset, then $f(B) \times f(B)$ is controlled. Thus $f(B)$ is relatively compact, and so is $f^{-1}(f(B))$. Therefore B is relatively compact. Since Y is coarsely connected, so is X . \square

The following is a typical example of the pullback coarse structure.

PROPOSITION 2.12. *Let X be a proper metric space. Let \mathcal{U} be a locally finite cover of X such that any element of \mathcal{U} has uniformly bounded diameter. Then (a geometric realization of) the nerve complex $|\mathcal{U}|$ has a canonical coarse structure which is proper and coarsely equivalent to X .*

PROOF. Since X is a proper metric space and \mathcal{U} is locally finite, $|\mathcal{U}|$ is locally compact second countable Hausdorff space. For each element $U \in \mathcal{U}$, we choose a point $x(U) \in U$. For each point $p \in |\mathcal{U}|$, we choose $U_p \in \mathcal{U}$ such that $p \in \text{st } U_p$, where $\text{st } U_p$ denotes the star of U_p . Then we define a map $f: |\mathcal{U}| \rightarrow X$ by $f(p) = x(U_p)$. Since \mathcal{U} is locally finite, the pullback $f^{-1}(K)$ of any compact set $K \subset X$ is relatively compact. Since each $U \in \mathcal{U}$

has uniformly bounded diameter, f is pseudocontinuous. Let $g: X \rightarrow |U|$ be a continuous map induced by a partition of unity. It is easy to see that $f \circ g$ is close to the identity. Thus the assertion follows from Proposition 2.9 and 2.11. \square

2.2. Higson compactification. Here we recall the definitions of the Higson compactification and coarse compactifications. For details, see [28] and [27].

DEFINITION 2.13. Let X be a proper coarse space and let V be a normed space. Let $f: X \rightarrow V$ be a bounded continuous function. We denote by $\mathbf{d}f$ the function

$$\mathbf{d}f(x, y) = f(y) - f(x): X \times X \rightarrow V.$$

We say that f is a Higson function, or, of vanishing variation, if for each controlled set E , the restriction of $\mathbf{d}f$ to E vanishes at infinity, that is, for any $\epsilon > 0$, there exists a bounded subset B such that for any $(x, y) \in E \setminus B \times B$, we have $\|\mathbf{d}f(x, y)\| < \epsilon$.

The bounded continuous \mathbb{C} -valued Higson functions on a proper coarse space X form a unital C^* -subalgebra of bounded continuous functions on X , which we denote $C_h(X)$. By the Gelfand-Naimark theory, $C_h(X)$ is isomorphic to a C^* -algebra of continuous functions on a compact Hausdorff space.

DEFINITION 2.14. The compactification hX of X characterized by the property $C(hX) = C_h(X)$ is called the Higson compactification. Its boundary $hX \setminus X$ is denoted νX , and is called the Higson corona of X .

The assignment $X \mapsto \nu X$ is a functor from the coarse category to the category of compact Hausdorff spaces. For details, see [28, Section 2.3] or [27, Section 5.1].

PROPOSITION 2.15 (Dranishnikov). *Let X and Y be proper metric spaces and let $f: X \rightarrow Y$ be a coarse embedding, that is, a coarse equivalence to the image. Then the induced map $\nu f: \nu X \rightarrow \nu Y$ is an embedding, thus we can regard νX as a subspace of νY .*

PROOF. The proposition follows immediately from [8, Theorem 1.4]. \square

DEFINITION 2.16. Let X be a proper coarse space. A corona of X is a pair (W, ζ) of a compact metrizable space W and a continuous map $\zeta: \nu X \rightarrow W$.

Let X be a proper coarse space. Let (W, ζ) be a corona of X . We consider the disjoint union $X \sqcup W$. We equip $X \sqcup W$ with the final topology with respect to the map

$\text{id} \sqcup \zeta: hX \rightarrow X \sqcup W$, which we denote by $\bar{\zeta}$. Let $X \cup_{\zeta} W$ denote the space $X \sqcup W$ with this topology. By the construction, we see that $X \cup_{\zeta} W$ is compact.

Next, we construct a compact Hausdorff space using functional analysis. The continuous map ζ induces a homomorphism $\zeta^*: C(W) \rightarrow C(\nu X)$. Then the image $\zeta^*(C(W))$ is a C^* -subalgebra of $C(\nu X)$. Let

$$\pi: C_h(X) \rightarrow C_h(X)/C_0(X) \cong C(\nu X)$$

be the quotient map. Then the pullback $\pi^{-1}(\zeta^*(C(W)))$ is a C^* -subalgebra of $C_h(X)$. Set $A = \{(f, g) \in \pi^{-1}(\zeta^*(C(W))) \oplus C(W) : \pi(f) = \zeta^*(g)\}$. Then A is a unital commutative C^* -algebra which contain $C_0(X)$ as an ideal. By the Gelfand-Naimark theory, there exists a compact Hausdorff space Z and an embedding $i: X \rightarrow Z$ such that $C(Z) \cong A$. We identify X with $i(X)$.

PROPOSITION 2.17. *These two spaces $X \cup_{\zeta} W$ and Z are homeomorphic. Especially, $X \cup_{\zeta} W$ is a compact metrizable space. If ζ is surjective, X is dense in $X \cup_{\zeta} W$ and thus we call $X \cup_{\zeta} W$ a coarse compactification of X . We abbreviate $X \cup_{\zeta} W$ to $X \cup W$ for simplicity.*

PROOF. Let A be a C^* -algebra defined in the above. The inclusion $C_0(X) \hookrightarrow A$ is given by $f \mapsto (f, 0)$. We also have a surjection $A \rightarrow C(W)$, $(f, g) \mapsto g$. We consider the following diagram with two short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_0(X) & \longrightarrow & C(hX) & \longrightarrow & C(\nu X) \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & C_0(X) & \longrightarrow & C(Z) & \longrightarrow & C(W) \longrightarrow 0. \end{array}$$

Since $C(W)$ and $C_0(X)$ are separable, so is $C(Z)$. Thus Z is metrizable. The surjection $C(Z) \rightarrow C(W)$ induces an embedding $W \rightarrow Z$, so we identify W with its image in Z . Thus Z can be decomposed as $Z = X \cup W$. Let $\varphi: X \cup_{\zeta} W \rightarrow Z$ be the canonical bijection. Then we have a commutative diagram

$$\begin{array}{ccc} hX & & \\ \zeta \downarrow & \searrow & \\ X \cup_{\zeta} W & \xrightarrow{\varphi} & Z. \end{array}$$

Since the map $hX \rightarrow Z$ is continuous, so is φ . Therefore φ is homeomorphism. \square

The following notion is useful in the study of proper metric spaces and their coronae from the view point of the algebraic topology.

DEFINITION 2.18. Let X and Y be proper metric spaces and let (W, ζ) and (Z, ξ) be respectively coronae of X and Y . Let $f: X \rightarrow Y$ be a coarse map and let $\eta: W \rightarrow Z$ be a continuous map. We say that f *covers* η if there exists a discrete subset $X' \subset X$ such that the inclusion is a coarse equivalence and the restriction $f|_{X'}$ extends to a continuous map $f \cup \eta: X' \cup W \rightarrow Y \cup Z$.

REMARK 2.19. In the above setting, f covers η if and only if the following diagram is commutative

$$\begin{array}{ccc} \nu X & \xrightarrow{\nu f} & \nu Y \\ \zeta \downarrow & & \xi \downarrow \\ W & \xrightarrow{\eta} & Z. \end{array}$$

In the rest of the paper, whenever we consider a corona (W, ζ) of a proper metric space X , we assume that X is non-compact. In particular, neither νX nor W is empty.

3. GENERALIZED COARSE COHOMOLOGY THEORY

3.1. Axiom. The coarse category is a category whose objects are proper metric spaces and whose morphisms are close classes of coarse maps. The coarse cohomology [27], the coarse K -theory [9] and the K -theory of the reduced stable Higson corona [9] can be regarded as cohomology theories on the coarse category. In this section, we introduce a generalized coarse cohomology theory.

The following notion was introduced in [19] to state the Mayer-Vietoris principle for the coarse cohomology and the K -theory of the Roe algebra. Let X be a metric space and $A \subset M$ be a subspace. For $R > 0$, we denote by $\text{Pen}(A; R)$ the R -neighborhood $\{x \in X : d(x, A) \leq R\}$ of A .

DEFINITION 3.1. Let X be a proper metric space, and let A and B be closed subspaces with $X = A \cup B$. We say that $X = A \cup B$ is an ω -*excisive decomposition*, if for each $R > 0$ there exists some $S > 0$ such that

$$\text{Pen}(A; R) \cap \text{Pen}(B; R) \subset \text{Pen}(A \cap B; S).$$

Higson-Roe [17] introduced a notion of coarse homotopy. After that, they gave an alternative definition of coarse homotopy, which is a variant of Lipschitz homotopy. (For

Lipschitz homotopy, see [14, 1.C₃], [32, Definition 4.1] and [16, Definition 11.1].) Our definition is based on [18, Section 11] and [31, Definition 3.9].

DEFINITION 3.2. Let $f, g: X \rightarrow Y$ be coarse maps between proper metric spaces. We say that they are *coarsely homotopic* if there exists a metric subspace $Z = \{(x, t) : 1 \leq t \leq T_x\}$ of $X \times \mathbb{N}$ and a coarse map $h: Z \rightarrow Y$, such that

- (a) the map $x \mapsto T_x$ is bornologous,
- (b) $h(x, 1) = f(x)$, and
- (c) $h(x, T_x) = g(x)$.

Here \mathbb{N} is a set of positive integers and we equip $X \times \mathbb{N}$ with the l_1 -metric, that is, $d_{X \times \mathbb{N}}((x, n), (y, m)) := d_X(x, y) + |n - m|$ for $(x, n), (y, m) \in X \times \mathbb{N}$, where d_X is the metric on X .

Coarse homotopy is then an equivalence relation on coarse maps.

DEFINITION 3.3. A *generalized coarse cohomology theory* is a contravariant functor $MX^* = \{MX^p\}_{p \in \mathbb{Z}}$ from the coarse category to the category of \mathbb{Z} -graded Abelian groups, such that

- (i) for a proper metric space X , we have $MX^*(X \times \mathbb{N}) = 0$, and
- (ii) if $X = A \cup B$ is an ω -excisive decomposition, there exists a functorial long exact sequence, called a Mayer-Vietoris exact sequence,

$$\cdots \rightarrow MX^p(X) \rightarrow MX^p(A) \oplus MX^p(B) \rightarrow MX^p(A \cap B) \rightarrow MX^{p+1}(X) \rightarrow \cdots.$$

The following notion of coarsely flasque spaces is based on [31, Definition 3.6].

LEMMA 3.4. Let MX^* be a generalized coarse cohomology theory. Let X be a space with a proper metric d . Suppose that X is coarsely flasque, that is, there exists a coarse map $\phi: X \rightarrow X$ such that

- (a) ϕ is close to the identity;
- (b) for any bounded subset $K \subset X$, there exists $N_K \in \mathbb{N}$ such that for any $n \geq N_K$, $\phi^n(X) \cap K = \emptyset$;
- (c) for all $R > 0$, there exists $S > 0$ such that for all $n \in \mathbb{N}$ and all $x, y \in X$ with $d(x, y) < R$, we have $(d(\phi^n(x), \phi^n(y))) < S$.

Then $MX^*(X) = 0$.

PROOF. We define a coarse map $\Phi: X \times \mathbb{N} \rightarrow X$ as $\Phi(x, n) = \phi^n(x)$. Then we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ & \searrow & \nearrow \Phi \\ & X \times \mathbb{N} & \end{array}.$$

Here $X \hookrightarrow X \times \mathbb{N}$ is the inclusion into $X \times \{1\}$. By axiom (i), the induced map $\phi^*: MX^*(X) \rightarrow MX^*(X)$ factors through zero. Since ϕ is close to the identity, $MX^*(X) = 0$. \square

The following coarse homotopy invariance follows from a standard argument using Mayer-Vietoris axiom (ii) and Lemma 3.4. (See [18, Proposition 12.4.12] and [31, Theorem 4.3.12.]).

PROPOSITION 3.5. *If two coarse maps $f, g: X \rightarrow Y$ are coarsely homotopic, the induced maps f^* and g^* are equal.*

The anti-Čech system is introduced in [27, Section 3] to relate the coarse cohomology to the Čech cohomology. It is also used in [17] to formulate a coarse homology theory.

DEFINITION 3.6. Let X be a metric space. Let $\mathcal{U}(1), \mathcal{U}(2), \dots$ be a sequence of locally finite covers of X . We say that they form an anti-Čech system if there exists a sequence of real numbers $R_n \rightarrow \infty$ such that for all n ,

- (a) each set $U \in \mathcal{U}(n)$ has diameter less than or equal to R_n , and
- (b) the covering $\mathcal{U}(n+1)$ has a Lebesgue number δ_{n+1} greater than or equal to R_n , that is, any set of diameter less than or equal to δ_{n+1} is contained in some element of $\mathcal{U}(n+1)$.

These conditions imply that for each n , there exists a map $\varphi_n: \mathcal{U}(n) \rightarrow \mathcal{U}(n+1)$ such that $U \subset \varphi_n(U)$ for all $U \in \mathcal{U}(n)$. We call φ_n a coarsening map. We remark that this map is called refining map in the context of Čech cohomology theory. A coarsening map φ_n induces a proper simplicial map $|\mathcal{U}(n)| \rightarrow |\mathcal{U}(n+1)|$ of the nerve complexes, which we also denote by the same symbol φ_n and also call a coarsening map.

Now we recall the definition of generalized cohomology theory on the category of locally compact and second countable Hausdorff spaces, which we abbreviate to LCSH. (See [18, Section 7.1] for LCSH.) A generalized cohomology theory on LCSH is a contravariant functor $M^* = \{M^p\}$ from LCSH to the category of \mathbb{Z} -graded Abelian groups such that

- (a) M^* is a homotopy functors, and
- (b) if $W \subset X$ is a closed subset, there is a functorial long exact sequence

$$\cdots \rightarrow M^p(X \setminus W) \rightarrow M^p(X) \rightarrow M^p(W) \xrightarrow{\partial} M^{p+1}(X \setminus W) \rightarrow \cdots.$$

Examples of such cohomology theories are K -theory $K^*(-)$ and the compactly supported Alexander-Spanier cohomology $H_c^*(-)$. These cohomology theories satisfy the continuity property

- (c) for a projective limit $X = \varprojlim X_n$ of locally compact second countable Hausdorff spaces, we have $M^*(X) \cong \varinjlim M^*(X_n)$.

Let W be a compact second countable Hausdorff space. Then the constant map $\pi_W: W \rightarrow \{*\}$ is proper, where $\{*\}$ is a one point space. The reduced M -cohomology of W , denoted by $\tilde{M}^*(W)$, is defined as the cokernel of π_W^* .

Let X be a proper metric space and let (W, ζ) be a corona of X . Let $\partial: M^p(W) \rightarrow M^{p+1}(X)$ be a boundary homomorphism of the long exact sequence for $W \subset X \cup_\zeta W$. Let $\pi_W: W \rightarrow \{*\}$ be a constant map. Since π_W factors through $X \cup_\zeta W \rightarrow \{*\}$, the image $\pi_W^*(M^p(\{*\}))$ lies on the kernel of ∂ . Thus we have a boundary homomorphism $\partial: \tilde{M}^p(W) \rightarrow M^{p+1}(X)$.

DEFINITION 3.7. Let $M^* = \{M^p\}_{p \in \mathbb{Z}}$ be a generalized cohomology theory on locally compact and second countable Hausdorff spaces. We say that a generalized coarse cohomology theory MX^* is a *coarsening* of M^* if MX^* satisfies the following:

- (iii) For a proper metric space X , there exists a *character map* $c: MX^*(X) \rightarrow M^*(X)$, which is an isomorphism if X is uniformly contractible and has bounded geometry. It is compatible with Mayer-Vietoris exact sequences of MX^* and M^* for ω -excisive decompositions.
- (iv) Let $\{\mathcal{U}_n\}$ be an anti-Čech system of a proper metric space X . There exists a functorial short exact sequence

$$0 \rightarrow \varprojlim^1 M^{q-1}(|\mathcal{U}_n|) \rightarrow MX^q(X) \xrightarrow{\theta} \varprojlim M^q(|\mathcal{U}_n|) \rightarrow 0.$$

Moreover, the composite of θ and a canonical map $\lambda: \varprojlim M^q(|\mathcal{U}_n|) \rightarrow M^q(X)$ is equal to the character map, where λ is given by a partition of unity. We call this a Milnor exact sequence.

- (v) Let (W, ζ) be a corona of X . Then there exists a *transgression* map $T_W: \tilde{M}^{q-1}(W) \rightarrow MX^q(X)$ such that $c \circ T_W = \partial$, here $\partial: \tilde{M}^{p-1}(W) \rightarrow M^p(X)$ is the boundary

homomorphism. The transgression map is natural in the following sense. For proper metric spaces X and Y , and for coronae (W, ζ) and (Z, ξ) respectively of X and Y , if a coarse map $f: X \rightarrow Y$ covers a continuous map $\eta: W \rightarrow Z$, then the following is commutative.

$$\begin{array}{ccc} \tilde{M}^{q-1}(Z) & \xrightarrow{\eta^*} & \tilde{M}^{q-1}(W) \\ \downarrow T_Z & & \downarrow T_W \\ MX^q(Y) & \xrightarrow{f^*} & MX^q(X). \end{array}$$

PROPOSITION 3.8. *The coarse cohomology $HX^*(-)$, the coarse K -theory $KX^*(-)$ and the K -theory of the reduced stable Higson corona $K_*(\mathfrak{c}^r(-))$ are generalized coarse cohomology theories. Especially, $KX^*(-)$ and $HX^*(-)$ are respectively the coarsening of the K -theory and the compactly supported Alexander-Spanier cohomology.*

PROOF. The statements for HX^* are proved in [27], those for $K_*(\mathfrak{c}^r(-))$ are proved in [9] and [31]. See Proposition 4.9. The statements for KX^* are proved in Section 4. \square

3.2. Coarse homology theories. Generalized coarse homology theories are formulated similarly, but we omit the detail. We remark that for a generalized homology theory M_* on LCSH, we have a generalized coarse homology theory MX_* by defining $MX_*(X) := \varinjlim M_*(|\mathcal{U}(j)|)$ where X is a proper metric space and $\{\mathcal{U}(j)\}_{j \in \mathbb{N}}$ is an anti-Čech system of X . (See [17, Section 2].) We say that MX_* is a coarsening of M_* . Using a partition of unity, we can define the coarsening map $c: M_*(X) \rightarrow MX_*(X)$. If X is uniformly contractible and has bounded geometry, the coarsening map c is an isomorphism. Emerson-Mayer proved a similar statement for coarse K -theory. (See [9, Theorem 4.8].) Their proof also works for MX_* . We remark that this statement is first proved in [17, Proposition 3.8] under an additional assumption that X is a simplicial complex with a spherical metric.

The transgression map is constructed as follows. Let X be a proper metric space and let (W, ζ) be a corona of X . Let $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ be an anti-Čech system of X . Since the nerve complex $|\mathcal{U}_n|$ is coarsely equivalent to X (Proposition 2.12), the pair (W, ζ) is also a corona of $|\mathcal{U}_n|$ and we obtain a compact space $|\mathcal{U}_n| \cup W$. A long exact sequence ([18, Definition 7.1.1]) for $W \subset |\mathcal{U}_n| \cup W$ defines the boundary homomorphism $\partial: M_*(|\mathcal{U}_n|) \rightarrow \tilde{M}_{*-1}(W)$. Here $\tilde{M}_*(W)$ is the reduced M -homology of W defined as the kernel of π_{W*} , where $\pi_W: Y \rightarrow \{*\}$ is a constant map. By taking the inductive limit,

we obtain $T_W: MX_*(X) \rightarrow \tilde{M}_{*-1}(W)$. From the construction, it is easy to see that the transgression map is natural in the obvious sense.

The K -theory of the Roe-algebra, the coarse K -homology are generalized coarse homology theories and the coarse K -homology is the coarsening of the K -homology. See [17], [19] and [18].

4. THE COARSE K -THEORY

4.1. The coarse K -theory. In this section we see that the coarse K -theory $KX^*(-)$ is a generalized coarse cohomology theory and is the coarsening of the K -theory $K^*(-)$ in the sense of the previous section. Originally, $KX^*(-)$ is defined and studied by Emerson-Meyer [9, Section 4]. We introduce a definition of $KX^*(-)$ by a slightly different manner, but we confirm that they are compatible. The original definition uses the Rips complex, while ours uses the anti-Čech system, which is more flexible and essentially used in the proof of Proposition 6.8.

Let X be a space with a proper metric d . Suppose that $\{\mathcal{U}(k)\}_{k \in \mathbb{N}}$ is an anti-Čech system of X with uniformly bounded diameter $R_k \rightarrow \infty$ and Lebesgue numbers $\delta_k \geq R_{k-1}$ of $\mathcal{U}(k)$.

For each $k \in \mathbb{N}$, we fix a coarsening map $\psi_{k,k+1}: |\mathcal{U}(k)| \rightarrow |\mathcal{U}(k+1)|$. We put $\psi_{k,l} := \psi_{l-1,l} \circ \cdots \circ \psi_{k,k+1}$ for each $k \in \mathbb{N}$ and $l \in \mathbb{N}$ with $k \leq l-1$ and we also call them coarsening maps. We denote the inductive limit by \mathcal{X} , which depends on choice of $\psi_{k,k+1}$. Also we denote the canonical map $|\mathcal{U}(k)| \rightarrow \mathcal{X}$ by $\psi_{k,\infty}$ for each $k \in \mathbb{N}$. We put

$$C_0(\mathcal{X}) := \{f: \mathcal{X} \rightarrow \mathbb{C} \mid f \circ \psi_{k,\infty} \in C_0(|\mathcal{U}(k)|) \text{ for any } k \in \mathbb{N}\}$$

and we identify it with the projective limit of $\{C_0(|\mathcal{U}(k)|)\}_{k \in \mathbb{N}}$. This is a σ - C^* -algebra. Now we define $KX^*(X)$ as $RK_*(C_0(\mathcal{X}))$. Here $RK_*(-)$ is a representable K -theory of σ - C^* -algebras [25]. We abbreviate $RK_*(C_0(\mathcal{X}))$ to $RK^*(\mathcal{X})$.

We remark that by Phillips [25], there exists an exact sequence, called a Milnor exact sequence,

$$(4) \quad 0 \rightarrow \varprojlim^1 K_{p+1}(C_0(|\mathcal{U}(k)|)) \rightarrow RK_p(C_0(\mathcal{X})) \rightarrow \varprojlim K_p(C_0(|\mathcal{U}(k)|)) \rightarrow 0.$$

See also Appendix B.

LEMMA 4.1. *Under the above setting, there exists an anti-Čech system $\{\mathcal{U}'(k)\}$ such that a coarsening map $\psi'_k: \mathcal{U}'(k) \rightarrow \mathcal{U}'(k+1)$ is injective for each $k \in \mathbb{N}$ and $RK^*(\mathcal{X}) \cong RK^*(\varinjlim \mathcal{U}'(k))$.*

PROOF. We take a copy $\mathcal{U}_i(k)$ of $\mathcal{U}(k)$ parameterized by $i \in \mathbb{N}$. Then $\bigcup_{i \in \mathbb{N}} \mathcal{U}_i(k)$ is a cover of X , but it is not locally finite. The identification between $\mathcal{U}_i(k)$ and $\mathcal{U}(k)$ define the surjection $P_k : \bigcup_{i \in \mathbb{N}} \mathcal{U}_i(k) \rightarrow \mathcal{U}(k)$. Then we can take an anti-Čech system $\{\mathcal{U}'(k)\}$ of X and proper injective simplicial map $\psi'_{k,k+1} : |\mathcal{U}'(k)| \rightarrow |\mathcal{U}'(k+1)|$ satisfying $\mathcal{U}'(1) = \mathcal{U}_1(1)$, $\mathcal{U}_1(k) \subset \mathcal{U}'(k) \subset \bigcup_{i \in \mathbb{N}} \mathcal{U}_i(k)$ and the following commutative diagram:

$$\begin{array}{ccccccc} \mathcal{U}'(1) & \xrightarrow{\psi'_{1,2}} & \mathcal{U}'(2) & \xrightarrow{\psi'_{2,3}} & \mathcal{U}'(3) & \xrightarrow{\psi'_{3,4}} & \dots \\ \downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & \\ \mathcal{U}(1) & \xrightarrow{\psi_{1,2}} & \mathcal{U}(2) & \xrightarrow{\psi_{2,3}} & \mathcal{U}(3) & \xrightarrow{\psi_{3,4}} & \dots, \end{array}$$

where p_k is a proper surjective simplicial map induced by P_k of the restriction on $|\mathcal{U}'(k)|$. For each k , we choose a section $e_k : \mathcal{U}(k) \rightarrow \mathcal{U}'(k)$ of p_k . Then we have the following commutative diagram:

$$\begin{array}{ccccccc} \mathcal{U}'(1) & \xrightarrow{\psi'_{1,2}} & \mathcal{U}'(2) & \xrightarrow{\psi'_{2,3}} & \mathcal{U}'(3) & \xrightarrow{\psi'_{3,4}} & \dots \\ \downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & \\ \mathcal{U}(1) & \xrightarrow{\psi_{1,2}} & \mathcal{U}(2) & \xrightarrow{\psi_{2,3}} & \mathcal{U}(3) & \xrightarrow{\psi_{3,4}} & \dots \\ \downarrow e_1 & & \downarrow e_2 & & \downarrow e_3 & & \\ \mathcal{U}'(1) & \xrightarrow{e_2 \circ \psi_{1,2} \circ p_1} & \mathcal{U}'(2) & \xrightarrow{e_3 \circ \psi_{2,3} \circ p_2} & \mathcal{U}'(3) & \xrightarrow{e_4 \circ \psi_{3,4} \circ p_3} & \dots \\ \downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & \\ \mathcal{U}(1) & \xrightarrow{\psi_{1,2}} & \mathcal{U}(2) & \xrightarrow{\psi_{2,3}} & \mathcal{U}(3) & \xrightarrow{\psi_{3,4}} & \dots. \end{array}$$

Note that the inductive limits of the second line and the forth line are \mathcal{X} . We denote by \mathcal{X}'' and \mathcal{X}''' , respectively, the inductive limits of the first line and the third line. Since every $p_k \circ e_k$ are identity maps, $(\varinjlim e_k)^* : RK_*(\mathcal{X}''') \rightarrow RK_*(\mathcal{X})$ is surjective. The Milnor exact sequence and its functoriality imply the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varprojlim^1 K^{*-1}(|\mathcal{U}'(k)|) & \longrightarrow & RK^*(\mathcal{X}'') & \longrightarrow & \varprojlim K^*(|\mathcal{U}'(k)|) \longrightarrow 0 \\ & & \uparrow \varprojlim^1 (e_k \circ p_k)^{*-1} & & \uparrow (\varinjlim e_k \circ \varinjlim p_k)^* & & \uparrow \varprojlim (e_k \circ p_k)^* \\ 0 & \longrightarrow & \varprojlim^1 K^{*-1}(|\mathcal{U}(k)|) & \longrightarrow & RK^*(\mathcal{X}''') & \longrightarrow & \varprojlim K^*(|\mathcal{U}(k)|) \longrightarrow 0. \end{array}$$

Since $e_k \circ p_k$ is contiguous to the identity map, $(\varinjlim e_k \circ \varinjlim p_k)^*$ is an isomorphism by the five lemma, and thus $(\varinjlim e_k)^*$ is injective. Hence $(\varinjlim p_k)^* : RK^*(\mathcal{X}) \rightarrow RK^*(\mathcal{X}'')$ is an isomorphism. \square

PROPOSITION 4.2. $KX^*(X)$ is well-defined, that is, this is independent of the choice of the anti-Čech system $\{\mathcal{U}(k)\}_{k \in \mathbb{N}}$ and the coarsening maps $\{\psi_{k,l}\}_{k \leq l}$.

PROOF. Let $\{\mathcal{U}(k)\}_{k \in \mathbb{N}}$ be an anti-Čech system and let $\{\psi_{k,l}\}_{k \leq l}$ be coarsening maps. By Lemma 4.1, we can assume that $\psi_{k,l}$ is injective. We denote by \mathcal{X} the injective limit of $\{\mathcal{U}(k)\}$.

We compare $\{\mathcal{U}(k)\}$ with a special kind of an anti-Čech system of X defined as follows. We take a subset Z of X and a constant $C > 0$ such that $\text{Pen}(Z, C) = X$ and $d(x, y) > 1$ for any $x, y \in Z$ with $x \neq y$. The existence of such a subset follows from Zorn's lemma. (See [27, Lemma 3.15].) We call Z a C -dense uniformly discrete subset of X . For each $k \in \mathbb{N}$, put $\mathcal{U}_{Z,C}(k) := \{\text{Pen}(z, (k+1)C) \subset X \mid z \in Z\}$ which is a locally finite cover of X since X is proper. For each $k \in \mathbb{N}$, diameter of any element of $\mathcal{U}_{Z,C}(k)$ is at most $2(k+1)C$ and the Lebesgue number of $\mathcal{U}_{Z,C}(k)$ is at least kC . Hence $\{\mathcal{U}_{Z,C}(k)\}_{k \in \mathbb{N}}$ is an anti-Čech system of X . We have a proper simplicial map $\iota_{k,l} : |\mathcal{U}_{Z,C}(k)| \rightarrow |\mathcal{U}_{Z,C}(l)|$ induced by $\mathcal{U}_{Z,C}(k) \ni \text{Pen}(z, (k+1)C) \mapsto \text{Pen}(z, (l+1)C) \in \mathcal{U}_{Z,C}(l)$ for each $k \in \mathbb{N}$ and $l \in \mathbb{N}$ with $k \leq l$. We denote the inductive limit by $\mathcal{X}_{Z,C}$. Also we denote the induced map $|\mathcal{U}_{Z,C}(k)| \rightarrow \mathcal{X}_{Z,C}$ by $\iota_{k,\infty}$ for each $k \in \mathbb{N}$. Note that $\iota_{k,l}$ is injective for any $k \in \mathbb{N}$ and $l \in \mathbb{N} \cup \{\infty\}$ with $k \leq l$.

We prove that $RK^*(\mathcal{X})$ and $RK^*(\mathcal{X}_{Z,C})$ are canonically isomorphic. Then we have the desired conclusion. We take an increasing sequence $\{k_j \in \mathbb{N}\}$ such that for each j , the cover $\mathcal{U}(j)$ is a refinement of $\mathcal{U}_{Z,C}(k_j)$. Then for each $j \in \mathbb{N}$, we can choose a coarsening map $f_j : \mathcal{U}(j) \rightarrow \mathcal{U}_{Z,C}(k_j)$ such that the following diagram:

$$\begin{array}{ccccc} |\mathcal{U}(1)| & \xrightarrow{\psi_{1,2}} & |\mathcal{U}(2)| & \xrightarrow{\psi_{2,3}} & \cdots \\ \downarrow f_1 & & \downarrow f_2 & & \\ |\mathcal{U}_{Z,C}(k_1)| & \xrightarrow{\iota_{k_1,k_2}} & |\mathcal{U}_{Z,C}(k_2)| & \xrightarrow{\iota_{k_2,k_3}} & \cdots \end{array}$$

is commutative without arranging any maps in both horizontal lines.

Next, we take an increasing sequence $\{k'_j \in \mathbb{N}\}$ such that for each j , $\mathcal{U}_{Z,C}(k_j)$ and $\mathcal{U}(k'_j)$ are respectively refinement of $\mathcal{U}(k'_j)$ and $\mathcal{U}(k'_{j+1})$. Then we can choose coarsening maps $g_j : \mathcal{U}_{Z,C}(k_j) \rightarrow \mathcal{U}(k'_j)$ and $\psi'_{k'_j, k'_{j+1}} : \mathcal{U}(k'_j) \rightarrow \mathcal{U}(k'_{j+1})$ such that the following diagram:

$$\begin{array}{ccccc} |\mathcal{U}_{Z,C}(k_1)| & \xrightarrow{\iota_{k_1,k_2}} & |\mathcal{U}_{Z,C}(k_2)| & \xrightarrow{\iota_{k_2,k_3}} & \cdots \\ \downarrow g_1 & & \downarrow g_2 & & \\ |\mathcal{U}(k'_1)| & \xrightarrow{\psi'_{k'_1,k'_2}} & |\mathcal{U}(k'_2)| & \xrightarrow{\psi'_{k'_2,k'_3}} & \cdots \end{array}$$

is commutative. We note that $\psi'_{k'_j, k'_{j+1}}$ is contiguous to $\psi_{k'_j, k'_{j+1}}$ and that $g_j \circ f_j$ is contiguous to ψ_{j, k'_j} . We denote by \mathcal{X}' the inductive limit of the second horizontal line. We remark that there are no canonical map from \mathcal{X}' to \mathcal{X} in general.

Again, we take an increasing sequence $\{k''_j \in \mathbb{N}\}$ such that for each j , covers $\mathcal{U}(k'_j)$ and $\mathcal{U}_{Z,C}(k''_j)$ are respectively refinements of $\mathcal{U}_{Z,C}(k''_j)$ and $\mathcal{U}_{Z,C}(k''_{j+1})$. Then we can choose coarsening maps $h_j : \mathcal{U}(k'_j) \rightarrow \mathcal{U}_{Z,C}(k''_j)$ and $\iota'_{k''_j, k''_{j+1}} : \mathcal{U}_{Z,C}(k''_j) \rightarrow \mathcal{U}_{Z,C}(k''_{j+1})$ such that the following diagram:

$$\begin{array}{ccccc} |\mathcal{U}(k'_1)| & \xrightarrow{\psi'_{k'_1, k'_2}} & |\mathcal{U}(k'_2)| & \xrightarrow{\psi'_{k'_2, k'_3}} & \dots \\ \downarrow h_1 & & \downarrow h_2 & & \\ |\mathcal{U}_{Z,C}(k''_1)| & \xrightarrow{\iota'_{k''_1, k''_2}} & |\mathcal{U}_{Z,C}(k''_2)| & \xrightarrow{\iota'_{k''_2, k''_3}} & \dots \end{array}$$

is commutative. We note that $\iota'_{k''_j, k''_{j+1}}$ is contiguous to $\iota_{k''_j, k''_{j+1}}$ and that $h_j \circ g_j$ is contiguous to ι_{k_j, k'_j} . We denote by $\mathcal{X}'_{Z,C}$ the inductive limit of the second horizontal line. We remark that there are no canonical map from $\mathcal{X}'_{Z,C}$ to $\mathcal{X}_{Z,C}$ in general.

Now we have a sequence of maps

$$\mathcal{X} \xrightarrow{f_\infty} \mathcal{X}_{Z,C} \xrightarrow{g_\infty} \mathcal{X}' \xrightarrow{h_\infty} \mathcal{X}'_{Z,C},$$

where we put $f_\infty := \varinjlim f_j$, $g_\infty := \varinjlim g_j$ and $h_\infty := \varinjlim h_j$. We prove that all maps induce isomorphisms of representable K -theory. Indeed we show that $g_\infty \circ f_\infty$ and $h_\infty \circ g_\infty$ induce isomorphisms of their representable K -theory.

We discuss only on the map $g_\infty \circ f_\infty$, since we can treat $h_\infty \circ g_\infty$ by the same way. We consider the following commutative diagram:

$$\begin{array}{ccccc} |\mathcal{U}(k'_1)| & \xrightarrow{\psi_{k'_1, k'_2}} & |\mathcal{U}(k'_2)| & \xrightarrow{\psi_{k'_2, k'_3}} & \dots \\ \psi_{1, k'_1} \uparrow & & \psi_{2, k'_2} \uparrow & & \\ |\mathcal{U}(1)| & \xrightarrow{\psi_{1, 2}} & |\mathcal{U}(2)| & \xrightarrow{\psi_{2, 3}} & \dots \\ g_1 \circ f_1 \downarrow & & g_2 \circ f_2 \downarrow & & \\ |\mathcal{U}(k'_1)| & \xrightarrow{\psi'_{k'_1, k'_2}} & |\mathcal{U}(k'_2)| & \xrightarrow{\psi'_{k'_2, k'_3}} & \dots \end{array}$$

The inductive limit of the first line is identified with that of the second line by the induced map $\varinjlim \psi_{j, k'_j}$. Thus we also denote by \mathcal{X} the inductive limit of the first line. By Milnor exact sequence (4) and its functoriality (see [25, Theorem 5.8 (5)] and also Proposition

B.1), we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \varprojlim^1 K^{*-1}(|\mathcal{U}(k'_j)|) & \longrightarrow & RK^*(\mathcal{X}) & \xrightarrow{\varprojlim \psi_{k'_j, \infty}} & \varprojlim K^*(|\mathcal{U}(k'_j)|) \longrightarrow 0 \\
& & \varprojlim^1 (\psi_{j, k'_j})^{*-1} \downarrow & & (\varinjlim \psi_{j, k'_j})^* \downarrow & & \downarrow \varprojlim (\psi_{j, k'_j})^* \\
0 & \longrightarrow & \varprojlim^1 K^{*-1}(|\mathcal{U}(j)|) & \longrightarrow & RK^*(\mathcal{X}) & \xrightarrow{\varinjlim \psi_{j, \infty}} & \varprojlim K^*(|\mathcal{U}(j)|) \longrightarrow 0 \\
& & \varprojlim^1 (g_j \circ f_j)^{*-1} \uparrow & & (g_\infty \circ f_\infty)^* \uparrow & & \uparrow \varprojlim (g_j \circ f_j)^* \\
0 & \longrightarrow & \varprojlim^1 K^{*-1}(|\mathcal{U}(k'_j)|) & \longrightarrow & RK^*(\mathcal{X}') & \xrightarrow{\varprojlim \psi'_{k'_j, \infty}} & \varprojlim K^*(|\mathcal{U}(k'_j)|) \longrightarrow 0.
\end{array}$$

Since $\varinjlim \psi_{j, k'_j} : \mathcal{X} \rightarrow \mathcal{X}$ is the identity map, $(\varinjlim \psi_{j, k'_j})^*$ is an isomorphism. Also $\varprojlim (\psi_{j, k'_j})^*$ is an isomorphism. Thus so is $\varprojlim^1 (\psi_{j, k'_j})^{*-1}$ by the five lemma. Since $g_j \circ f_j$ is contiguous to ψ_{j, k'_j} , both $\varprojlim^1 (g_j \circ f_j)^{*-1}$ and $\varprojlim (g_j \circ f_j)^*$ are isomorphisms, thus so is $(g_\infty \circ f_\infty)^*$. \square

By the definition and Milnor exact sequence (4), $KX^*(-)$ satisfies axiom (iv).

Suppose we have a proper metric space Y and a coarse map $f : X \rightarrow Y$. We take an anti-Čech system $\{\mathcal{V}(k)\}_{k \in \mathbb{N}}$ of Y . We take an increasing sequence $\{k_j \in \mathbb{N}\}$ such that for each j , the covers $\mathcal{U}(j)$ and $\mathcal{V}(k_j)$ are respectively refinement of $\mathcal{U}(k_j)$ and $\mathcal{V}(k_{j+1})$. Then we can choose a map $f_j : \mathcal{U}(j) \rightarrow \mathcal{V}(k_j)$ and $\phi_{k_j, k_{j+1}} : \mathcal{V}(k_j) \rightarrow \mathcal{V}(k_{j+1})$ such that $f(U) \subset f_j(U)$ for any $U \in \mathcal{U}(j)$ and the following diagram is commutative.

$$\begin{array}{ccccc}
|\mathcal{U}(1)| & \xrightarrow{\psi_{1,2}} & |\mathcal{U}(2)| & \xrightarrow{\psi_{2,3}} & \dots \\
\downarrow f_1 & & \downarrow f_2 & & \\
|\mathcal{V}(k_1)| & \xrightarrow{\phi_{k_1, k_2}} & |\mathcal{V}(k_2)| & \xrightarrow{\phi_{k_2, k_3}} & \dots
\end{array}$$

This induces a homomorphism $f^* : KX^*(Y) \rightarrow KX^*(X)$, which does not depend on the choice of anti-Čech systems, the maps f_j and $\phi_{k_j, k_{j+1}}$. Let $g : X \rightarrow Y$ be another coarse map which is close to f . Then we have $f^* = g^*$. These facts can be proved by the similar arguments with the proof of Proposition 4.2, so we omit the details.

Let Z be a C -dense uniformly discrete subset of X . Then $KX^*(Z)$ coincides with the coarse K -theory of X defined by Emerson-Mayer[9]. Since Z and X are coarsely equivalent, we have $KX^*(Z) \cong KX^*(X)$. Hence Emerson-Meyer's definition and ours are compatible.

LEMMA 4.3. *The coarse K -theory satisfies axiom (i).*

PROOF. Let $\{\mathcal{U}(k)\}_{k \in \mathbb{N}}$ be an anti-Čech system of X . Let $\psi_k : \mathcal{U}(k) \rightarrow \mathcal{U}(k+1)$ denote a coarsening map. Set $\mathcal{V}(k) := \{U \times [n, n+k] : U \in \mathcal{U}(k), n \in \mathbb{N}\}$. Then $\{\mathcal{V}(k)\}$ forms

an anti-Čech system of $X \times \mathbb{N}$. For $k \in \mathbb{N}$ and $s \in \mathbb{N} \cup \{0\}$, we define a simplicial map $\phi_{k,s}: |\mathcal{V}(k)| \rightarrow |\mathcal{V}(k+1)|$ by

$$\phi_{k,s}(U \times [n, n+k]) := \begin{cases} \psi_k(U) \times [n, n+k+1] & \text{if } n > s, \\ \psi_k(U) \times [n+1, n+k+2] & \text{if } n \leq s \end{cases}$$

where $U \in \mathcal{U}(k)$. Since $\phi_{k,s}$ is contiguous to $\phi_{k,s+1}$, we have a proper homotopy

$$h_{k,s}(t): |\mathcal{V}(k)| \rightarrow |\mathcal{V}(k+1)|$$

between geometric realization of $\phi_{k,s}$ and $\phi_{k,s+1}$ where $t \in [s, s+1]$. Then we define a continuous proper map $H_k: |\mathcal{V}(k)| \times \mathbb{R}_{\geq 0} \rightarrow |\mathcal{V}(k+1)|$ by $H_k(x, t) = h_{k,s}(t)(x)$ where s is an integer satisfying $t \in [s, s+1]$. We remark that the restriction $H_k(-, 0)$ is a coarsening map $\phi_{k,0}$. Thus the induced map $\phi_{k,0}^*: K^*(|\mathcal{V}(k+1)|) \rightarrow K^*(|\mathcal{V}(k)|)$ factors through $K^*(|\mathcal{V}(k)| \times \mathbb{R}_{\geq 0}) = 0$, so $\varprojlim K^*(|\mathcal{V}(k)|) = \varprojlim^1 K^*(|\mathcal{V}(k)|) = 0$. Therefore $KX^*(X \times \mathbb{R}_{\geq 0}) = 0$. \square

We need the following lemma to show that $KX^*(-)$ satisfies axiom (ii).

LEMMA 4.4. *Let the following be a pullback diagram of σ - C^* -algebras:*

$$\begin{array}{ccc} P_k & \xrightarrow{g_{1,k}} & A_{1,k} \\ g_{2,k} \downarrow & & \downarrow f_{1,k} \\ A_{2,k} & \xrightarrow{f_{2,k}} & B_k, \end{array}$$

where we suppose that $f_{1,k}$ and $f_{2,k}$ are surjective for any $k \in \mathbb{N}$. Let $\Pi_k: P_{k+1} \rightarrow P_k$, $\pi_{1,k}: A_{1,k+1} \rightarrow A_{1,k}$, $\pi_{2,k}: A_{2,k+1} \rightarrow A_{2,k}$ and $\pi_k: B_{k+1} \rightarrow B_k$ be $*$ -homomorphisms. Suppose that the following diagram is commutative for every $k \in \mathbb{N}$

$$\begin{array}{ccccc} P_{k+1} & \xrightarrow{g_{1,k+1}} & A_{1,k+1} & & \\ \downarrow g_{2,k+1} & \searrow \Pi_k & \downarrow & \searrow \pi_{1,k} & \\ & P_k & \xrightarrow{g_{1,k}} & A_{1,k} & \\ & \downarrow & \downarrow f_{1,k+1} & \downarrow f_{1,k} & \\ A_{2,k+1} & \xrightarrow{f_{2,k+1}} & B_{k+1} & & \\ \downarrow \pi_{2,k} & \searrow g_{2,k} & \downarrow \pi_k & & \\ & A_{2,k} & \xrightarrow{f_{2,k}} & B_k. & \end{array}$$

Then we have the following Mayer-Vietoris exact sequence:

$$\longrightarrow RK_{*+1}(\varprojlim B_k) \longrightarrow RK_*(\varprojlim P_k) \longrightarrow RK_*(\varprojlim A_{1,k}) \oplus RK_*(\varprojlim A_{2,k}) \longrightarrow .$$

PROOF. We refer to Proof of [1, Theorem 21.2.2].

By taking projective limit, we have the following commutative diagram

$$\begin{array}{ccc} P_\infty := \varprojlim P_k & \xrightarrow[g_{2,\infty}]{:=\varprojlim g_{2,k}} & A_{1,\infty} := \varprojlim A_{1,k} \\ g_{1,\infty} \downarrow & & f_{1,\infty} \downarrow \\ A_{2,\infty} := \varprojlim A_{2,k} & \xrightarrow[f_{2,\infty}]{:=\varprojlim f_{2,k}} & B_\infty := \varprojlim B_k, \end{array}$$

which is not necessarily a pull-back diagram. Put for each $k \in \mathbb{N} \cup \{\infty\}$,

$$C_k := \{(h_{1,k}, h_{2,k}) \in C_0([0, 1]) \otimes A_{1,k} \oplus C_0([0, 1]) \otimes A_{2,k} \mid f_{1,k}(h_{1,k}(0)) = f_{2,k}(h_{2,k}(0))\}.$$

For a σ - C^* -algebra A , we denote by SA the suspension $C_0(0, 1) \otimes A$. For each $k \in \mathbb{N} \cup \{\infty\}$, there is a canonical map $\psi_k : C_k \rightarrow SB_k$ defined by

$$[\psi_k(h_{1,k}, h_{2,k})](t) := \begin{cases} f_{1,k}(h_{1,k}(1 - 2t)) & \text{for } t \leq \frac{1}{2} \\ f_{2,k}(h_{2,k}(2t - 1)) & \text{for } t \geq \frac{1}{2} \end{cases}.$$

Then we have the following commutative diagram where two horizontal sequences are both exact,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varprojlim^1 RK_{*+1}(C_k) & \longrightarrow & RK_*(\varprojlim C_k) & \longrightarrow & \varprojlim RK_*(C_k) \longrightarrow 0 \\ & & \downarrow \varprojlim^1 (\psi_k)_{*+1} & & \downarrow (\psi_\infty)_* & & \downarrow \varprojlim (\psi_k)_* \\ 0 & \longrightarrow & \varprojlim^1 RK_{*+1}(SB_k) & \longrightarrow & RK_*(S \varprojlim B_k) & \longrightarrow & \varprojlim RK_*(SB_k) \longrightarrow 0. \end{array}$$

Since $(\psi_k)_*$ is an isomorphism for each k , so is a map $(\psi_\infty)_*$.

We have the following

$$\begin{array}{ccccccc} 0 & \longrightarrow & SA_{1,k+1} \oplus SA_{2,k+1} & \longrightarrow & C_{k+1} & \longrightarrow & P_{k+1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & SA_{1,k} \oplus SA_{2,k} & \longrightarrow & C_k & \longrightarrow & P_k \longrightarrow 0. \end{array}$$

Here each horizontal sequence is exact. (See [23, Section 2].) Since the left vertical map is surjective by the given condition, we have an exact sequence

$$0 \rightarrow SA_{1,\infty} \oplus SA_{2,\infty} \rightarrow C_\infty \rightarrow P_\infty \rightarrow 0.$$

We define $\kappa_\infty : SA_{1,\infty} \oplus SA_{2,\infty} \rightarrow SB_\infty$ as the restriction of ψ_∞ . Then we have the following exact sequence

$$\begin{array}{ccccccc} \longrightarrow & RK_{*+1}(P_\infty) & \xrightarrow{\partial} & RK_*(SA_{1,\infty} \oplus SA_{2,\infty}) & \longrightarrow & RK_*(C_\infty) & \longrightarrow \\ & & & \cong \downarrow & \searrow (\kappa_\infty)_* & (\psi_\infty)_* \downarrow \cong & \\ & & & RK_*(SA_{1,\infty}) \oplus RK_*(SA_{2,\infty}) & & RK_*(SB_\infty) & \end{array}$$

This gives the desired exact sequence by $RK_{*+1}(-) \cong RK_*(S-)$. \square

Proof of Proposition 3.8 for $KX^(-)$.* We prove that $KX^*(-)$ satisfies axiom (ii). Let X be a space with a proper metric d . We take a C -dense uniformly discrete subset Z of X . We denote $\mathcal{U}_{Z,C}(k)$ in Proof of Claim 4.2 by $\mathcal{U}(k)$ in this proof. It is straightforward to show the following claim.

CLAIM 4.5. *Let $L \subset X$ be a closed subset. By restriction, we have an anti-Čech system $\{L \cap \mathcal{U}(k) := \{L \cap U \mid U \in \mathcal{U}(k)\}\}_{k \in \mathbb{N}}$ of L . Also we consider the subcomplex $|\mathcal{U}(k)^L|$ of $|\mathcal{U}(k)|$ completely spanned by $\mathcal{U}(k)^L := \{U \in \mathcal{U}(k) \mid L \cap U \neq \emptyset\}$ for each $k \in \mathbb{N}$. Then we have an injective proper simplicial map $|L \cap \mathcal{U}(k)| \hookrightarrow |\mathcal{U}(k)^L|$ induced by $L \cap \mathcal{U}(k) \ni L \cap U \mapsto U \in \mathcal{U}(k)^L$. This induces an isomorphism from $\varprojlim C_0(|\mathcal{U}(k)^L|)$ to $\varprojlim C_0(|L \cap \mathcal{U}(k)|)$ as σ - C^* -algebras and thus induces an isomorphism from $RK_*(\varprojlim C_0(|\mathcal{U}(k)^L|))$ to $RK_*(\varprojlim C_0(|L \cap \mathcal{U}(k)|))$*

Note that $KX^*(L) = RK_*(\varprojlim C_0(|L \cap \mathcal{U}(k)|))$ in the above.

Now we consider an ω -excisive decomposition $X = A \cup B$. Then $|\mathcal{U}(k)| = |\mathcal{U}(k)^A| \cup |\mathcal{U}(k)^B|$ is an excisive decomposition as simplicial complexes. Hence we have the following projective system of pull-back diagrams of C^* -algebras:

$$\begin{array}{ccc} C_0(|\mathcal{U}(k)|) & \longrightarrow & C_0(|\mathcal{U}(k)^B|) \\ \downarrow & & \downarrow \\ C_0(|\mathcal{U}(k)^A|) & \longrightarrow & C_0(|\mathcal{U}(k)^A| \cap |\mathcal{U}(k)^B|). \end{array}$$

Since $|\mathcal{U}(k)^L| \rightarrow |\mathcal{U}(k+1)^L|$ is injective for any closed subspace $L \subset X$, Lemma 4.4 implies the following exact sequence:

$$\begin{aligned} \cdots \rightarrow RK_*(\varprojlim C_0(|\mathcal{U}(k)^A|)) \oplus RK_*(\varprojlim C_0(|\mathcal{U}(k)^B|)) &\rightarrow RK_*(\varprojlim C_0(|\mathcal{U}(k)|)) \\ &\rightarrow RK_{*-1}(\varprojlim C_0(|\mathcal{U}(k)^A| \cap |\mathcal{U}(k)^B|)) \rightarrow \cdots \end{aligned}$$

It follows from Claim 4.5 that $KX^*(A)$, $KX^*(B)$ and $KX^*(X)$ are naturally isomorphic to $RK_*(\varprojlim C_0(|\mathcal{U}(k)^A|))$, $RK_*(\varprojlim C_0(|\mathcal{U}(k)^B|))$ and $RK_*(\varprojlim C_0(|\mathcal{U}(k)|))$, respectively.

Now we prove that $RK_*(\varprojlim C_0(|\mathcal{U}(k)^A| \cap |\mathcal{U}(k)^B|))$ is naturally isomorphic to $KX^*(A \cap B)$. We have a natural injection $|\mathcal{U}(k)^{A \cap B}| \hookrightarrow |\mathcal{U}(k)^A| \cap |\mathcal{U}(k)^B|$. Also we have $|\mathcal{U}(k)^A| \cap |\mathcal{U}(k)^B| \hookrightarrow |\mathcal{U}(k)^{\text{Pen}(A, 2(k+1)C) \cap \text{Pen}(B, 2(k+1)C)}|$. Since $X = A \cup B$ is an ω -excisive decomposition, there exists $k' \in \mathbb{N}$ such that $\text{Pen}(A, 2(k+1)C) \cap \text{Pen}(B, 2(k+1)C) \subset \text{Pen}(A \cap B, 2(k'+1)C)$. Hence we have $|\mathcal{U}(k)^{\text{Pen}(A, 2(k+1)C) \cap \text{Pen}(B, 2(k+1)C)}| \hookrightarrow |\mathcal{U}(k)^{\text{Pen}(A \cap B, 2(k'+1)C)}|$. Then we have $|\mathcal{U}(k)^{\text{Pen}(A \cap B, 2(k'+1)C)}| \hookrightarrow |\mathcal{U}((k+2k'+3)C)^{A \cap B}|$. By taking an increasing sequence $\{k_j \in \mathbb{N}\}_j$, we have the following commutative diagram:

$$\begin{array}{ccccccc}
 |\mathcal{U}(k_1)^{A \cap B}| & \longrightarrow & |\mathcal{U}(k_2)^{A \cap B}| & \longrightarrow & |\mathcal{U}(k_3)^{A \cap B}| & \longrightarrow & \dots \\
 \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\
 |\mathcal{U}(k_1)^A| \cap |\mathcal{U}(k_1)^B| & \longrightarrow & |\mathcal{U}(k_2)^A| \cap |\mathcal{U}(k_2)^B| & \longrightarrow & |\mathcal{U}(k_3)^A| \cap |\mathcal{U}(k_3)^B| & \longrightarrow & \dots
 \end{array}$$

This implies that $\varprojlim C_0(|\mathcal{U}(k_j)^{A \cap B}|) \cong \varprojlim C_0(|\mathcal{U}(k_j)^A| \cap |\mathcal{U}(k_j)^B|)$. By combining Claim 4.5, we have that $RK_*(\varprojlim C_0(|\mathcal{U}(k)^A| \cap |\mathcal{U}(k)^B|))$ is naturally isomorphic to $KX^*(A \cap B)$. Hence we have the desired exact sequence:

$$\dots \rightarrow KX^*(A) \oplus KX^*(B) \rightarrow KX^*(X) \rightarrow KX^{*+1}(A \cap B) \rightarrow \dots$$

We can easily confirm its functoriality.

Now we show that $KX^*(-)$ satisfies axiom (iii). We have a proper continuous map $X \rightarrow |\mathcal{U}(1)|$ by using partition of unity (see [17, Section 3]). Then we have a $*$ -homomorphism $\varprojlim C_0(|\mathcal{U}(k)|) \rightarrow C_0(X)$. This induces the character map $c : KX^*(X) \rightarrow K^*(X)$. It follows from Proof of the axiom (ii) that the character maps preserve Mayer-Vietoris sequences for ω -excisive decomposition. Also the character map for a uniformly contractible proper metric space with bounded geometry is an isomorphism by [9, Theorem 4.8]. We can confirm that this does not depend on the choice of partition of unity and so on.

Finally we show that $KX^*(-)$ satisfies axiom (v). We consider a proper continuous map $\epsilon : X \rightarrow |\mathcal{U}(1)|$ in the above. Then we can give a proper coarse structure on $|\mathcal{U}(k)|$ such that $\iota_{i,k} \circ \epsilon : X \rightarrow |\mathcal{U}(k)|$ is a coarse equivalence by using Proposition 2.9. Hence if W is a corona of X , then W is naturally a corona of $|\mathcal{U}(k)|$ for each $k \in \mathbb{N}$. We have the

following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_0(|\mathcal{U}(k+1)|) & \longrightarrow & C(|\mathcal{U}(k+1)| \cup W) & \longrightarrow & C(W) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow = \\
0 & \longrightarrow & C_0(|\mathcal{U}(k)|) & \longrightarrow & C(|\mathcal{U}(k)| \cup W) & \longrightarrow & C(W) \longrightarrow 0,
\end{array}$$

where we can assume that left vertical map is surjective without loss of generality. Hence we have

$$0 \longrightarrow \varprojlim C_0(|\mathcal{U}(k)|) \longrightarrow \varprojlim C(|\mathcal{U}(k)| \cup W) \longrightarrow C(W) \longrightarrow 0.$$

The map ϵ induces the following:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \varprojlim C_0(|\mathcal{U}(k)|) & \longrightarrow & \varprojlim C(|\mathcal{U}(k)| \cup W) & \longrightarrow & C(W) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow = \\
0 & \longrightarrow & C_0(X) & \longrightarrow & C(X \cup W) & \longrightarrow & C(W) \longrightarrow 0.
\end{array}$$

Since the inclusion $\mathbb{C} \rightarrow C(W)$ factors through $\varprojlim C(|\mathcal{U}(k)| \cup W) \rightarrow C(W)$, we have

$$\begin{array}{ccc}
\tilde{K}^{*-1}(W) & \xrightarrow{T_W} & KX^*(X) \\
& \searrow \partial & \downarrow c \\
& & K^*(X).
\end{array}$$

From the construction, it is easy to see that the transgression map is natural in the sense of axiom (v). \square

4.2. The coarse co-assembly map. Let X be a proper metric space. We denote by $B(\mathcal{H})$ the C^* -algebra of bounded linear operators on a separable infinite dimensional Hilbert space \mathcal{H} . We also denote by \mathfrak{K} the C^* -algebra of compact operators on \mathcal{H} .

DEFINITION 4.6 ([9]). We let $\bar{\mathfrak{c}}^r$ be the C^* -algebra of bounded continuous $B(\mathcal{H})$ -valued Higson function on X such that $f(x) - f(y) \in \mathfrak{K}$ for all $x, y \in X$. The quotient $\mathfrak{c}^r(X) := \bar{\mathfrak{c}}^r(X)/C_0(X, \mathfrak{K})$ is called the reduced stable Higson corona of X .

See [9, Definition 4.3] for the unreduced stable Higson corona.

PROPOSITION 4.7 ([9]). *The assignment $X \mapsto \mathfrak{c}^r(X)$ is a contravariant functor from the coarse category to the category of C^* -algebras.*

Let $\{\mathcal{U}_n\}$ be an anti-Čech system of X . We fix coarsening maps $|\mathcal{U}_n| \rightarrow |\mathcal{U}_{n+1}|$ and put $\mathcal{X} := \varinjlim |\mathcal{U}_n|$. Then we have canonical maps $\Psi_n: |U_n| \rightarrow \mathcal{X}$. We put

$$C_0(\mathcal{X}, \mathfrak{K}) := \{f: \mathcal{X} \rightarrow \mathfrak{K} : f \circ \Psi_n \in C_0(|U_n|, \mathfrak{K}) \text{ for all } n \in \mathbb{N}\};$$

$$\bar{\mathfrak{c}}^r(\mathcal{X}) := \{f: \mathcal{X} \rightarrow B(\mathcal{H}) : f \circ \Psi_n \in \bar{\mathfrak{c}}^r(|U_n|) \text{ for all } n \in \mathbb{N}\}.$$

Both of $C_0(\mathcal{X}, \mathfrak{K})$ and $\bar{\mathfrak{c}}^r(\mathcal{X})$ are σ - C^* -algebras. We have

$$C_0(\mathcal{X}, \mathfrak{K}) = \varprojlim C_0(|U_n|, \mathfrak{K}), \quad \bar{\mathfrak{c}}^r(\mathcal{X}) = \varprojlim \bar{\mathfrak{c}}^r(|U_n|).$$

Since coarsening maps $X \rightarrow |U_n|$ and $|U_n| \rightarrow |U_{n+1}|$ are coarse equivalences, Proposition 4.7 implies that the projective limit

$$\mathfrak{c}^r(\mathcal{X}) := \varprojlim \mathfrak{c}^r(|U_n|)$$

is again a C^* -algebra, which is isomorphic to $\mathfrak{c}^r(X)$. The following sequences of σ - C^* -algebras is exact ([9, Lemma 3.12]).

$$(5) \quad 0 \rightarrow C_0(\mathcal{X}, \mathfrak{K}) \rightarrow \bar{\mathfrak{c}}^r(\mathcal{X}) \rightarrow \mathfrak{c}^r(\mathcal{X}) \rightarrow 0.$$

DEFINITION 4.8 ([9]). Let X be a proper metric space. The coarse co-assembly map for X is the map

$$\mu^*: K_{*+1}(\mathfrak{c}^r(X)) \rightarrow KX^*(X)$$

that is obtained from the connecting map of the exact sequence (5).

PROPOSITION 4.9 (Emerson-Meyer, Willet). *The K -theory of the reduced stable Higson corona is a generalized coarse cohomology theory.*

PROOF. The axiom (i) follows from [9, Theorem 5.2.]. The axiom (ii) is proved in [31, Proposition 4.3.6]. \square

The Mayer-Vietoris exact sequences for both of $K_*(\mathfrak{c}^r(-))$ and $KX^*(-)$ come from the general notion of the Mayer-Vietoris exact sequence associated to a pull-back diagram of C^* -algebras. (See [1, Theorem 21.2.2].) Therefore, the connecting maps in both of these exact sequences and coarse co-assembly maps are naturally commutative. That is, for an ω -excisive decomposition $X = A \cup B$ of a proper metric space X , we have the following

commutative diagram,

$$\begin{array}{ccccccc}
\longrightarrow & K_{p+1}(\mathfrak{c}^r(X)) & \longrightarrow & K_{p+1}(\mathfrak{c}^r(A)) \oplus K_{p+1}(\mathfrak{c}^r(B)) & \longrightarrow & K_{p+1}(\mathfrak{c}^r(A \cap B)) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & KX^p(X) & \longrightarrow & KX^p(A) \oplus KX^p(B) & \longrightarrow & KX^p(A \cap B) & \longrightarrow
\end{array}$$

where both of horizontal sequences are exact and vertical maps are coarse co-assembly maps.

5. COARSE COHOMOLOGY OF HYPERBOLIC METRIC SPACES

In this section, we summarize the result of [26] and [17] from the view point of the coarse cohomology theories. Let M^* be the K -theory or the compactly supported Alexander-Spanier cohomology and let MX^* be its coarsening.

5.1. The transgression map of the open cone. Let Y be a compact subset of the unit sphere in a separable Hilbert space H . The open cone on Y , denoted $\mathcal{O}Y$, is the set of all non-negative multiples of points in Y . The closed cone $\mathcal{C}Y = \{tx \in H : t \in [0, 1], x \in Y\}$ is a compactification of $\mathcal{O}Y$ and Y is a corona of it. By axiom (v), there is a commutative diagram. (See also [27, Example 5.28].)

(6)

$$\begin{array}{ccc}
& & MX^q(\mathcal{O}Y) \\
& \nearrow T_Y & \downarrow c \\
\tilde{M}^{q-1}(Y) & & \\
& \searrow \partial & \\
& & M^q(\mathcal{O}Y)
\end{array}$$

Here T_Y is a transgression map and ∂ is the boundary map in the long exact cohomology sequence for $Y \subset \mathcal{C}Y$.

LEMMA 5.1. *The character map $c: MX^q(\mathcal{O}Y) \rightarrow M^q(\mathcal{O}Y)$ and the transgression map $T_Y: \tilde{M}^{q-1}(Y) \rightarrow MX^q(\mathcal{O}Y)$ are isomorphisms.*

PROOF. First, we consider a cohomology long exact sequence for $Y \subset \mathcal{C}Y$. Since $\mathcal{C}Y$ is homotopic to one point, the long exact sequence splits and we obtain

$$0 \rightarrow M^{q-1}(\mathcal{C}Y) \rightarrow M^{q-1}(Y) \xrightarrow{\partial} M^q(\mathcal{O}Y) \rightarrow 0.$$

Hence $\partial: \tilde{M}^{q-1}(Y) \rightarrow M^q(\mathcal{O}Y)$ is an isomorphism.

Next, let $\{\mathcal{U}_i\}$ be an anti-Čech system of $\mathcal{O}Y$ constructed in the proof of [17, Proposition 4.3] (see also [12, Appendix B]). Then it is shown that:

- Each $|\mathcal{U}_i|$ is equipped with a proper coarse structure which is coarsely equivalent to $\mathcal{O}Y$, so Y is also a corona of $|\mathcal{U}_i|$. Thus we have a coarse compactification $\overline{|\mathcal{U}_i|} := |\mathcal{U}_i| \cup Y$.
- The coarsening map $|\mathcal{U}_i| \rightarrow |\mathcal{U}_{i+1}|$ covers the identity on Y .
- The extended map $\overline{|\mathcal{U}_i|} \rightarrow \overline{|\mathcal{U}_{i+1}|}$ is nullhomotopic.

By the argument similar to the proof of [17, Proposition 4.3], we can show that the boundary map ∂ gives an isomorphism between $\tilde{M}^{q-1}(Y)$ and $\text{Im}[M^q(|\mathcal{U}_{i+1}|) \rightarrow M^q(|\mathcal{U}_i|)]$. This implies $\varprojlim^1 M^q(|\mathcal{U}_i|) = 0$ and $\tilde{M}^{q-1}(Y) \cong \varprojlim M^q(|\mathcal{U}_i|)$. Thus it follows from axiom (iv) that the character map $c: MX^q(\mathcal{O}Y) \rightarrow M^q(\mathcal{O}Y)$ is an isomorphism. Now the diagram (6) shows that the transgression map T_Y is an isomorphism. \square

5.2. Hyperbolic spaces. Let X be a proper geodesic space which is hyperbolic in the sense of Gromov. Roe [26] showed that the Gromov boundary of X , denoted by ∂X , is a corona of X . Higson-Roe [17] constructed a coarse map $\mathcal{O}(\partial X) \rightarrow X$ and showed that it is a coarse homotopy equivalence. Thus by coarse homotopy invariance, we have $MX^*(X) \cong MX^*(\mathcal{O}(\partial X))$. For details, see [17, Section 8] and [31, Section 4.7]. By the same reason, we have $K_*(\mathfrak{c}^r(X)) \cong K_*(\mathfrak{c}^r(\mathcal{O}(\partial X)))$. Willett [31, Section 4.5] showed that the coarse co-assembly map for the open cone $\mathcal{O}(\partial X)$ is an isomorphism. Therefore we have the following.

PROPOSITION 5.2. *Let X be a proper geodesic space which is hyperbolic in the sense of Gromov. Then the coarse co-assembly map $\mu^*: K_{*+1}(\mathfrak{c}^r(X)) \rightarrow KX^*(X)$ is an isomorphism.*

It is easy to see that the coarse map $\mathcal{O}(\partial X) \rightarrow X$ covers the identity on ∂X . Therefore, by Lemma 5.1, axiom (v) and coarse homotopy invariance, we have the followings.

COROLLARY 5.3. *Let X be a non-compact proper geodesic space which is hyperbolic in the sense of Gromov. The transgression maps*

$$T_{\partial X}: KX_*(X) \rightarrow \tilde{K}_{*-1}(\partial X);$$

$$T_{\partial X}: \tilde{K}^{*-1}(\partial X) \rightarrow KX^*(X);$$

$$T_{\partial X}: \tilde{H}^{*-1}(\partial X) \rightarrow HX^*(X).$$

are isomorphisms.

6. RELATIVELY HYPERBOLIC GROUPS

Let G be a finitely generated group with a finite family of infinite subgroups $\mathbb{P} = \{P_1, \dots, P_k\}$. Groves and Manning [15] introduced an *augmented space* on which G acts properly discontinuously by isometries. The augmented space characterizes hyperbolicity of G relative to \mathbb{P} . We review the construction and show that there exists a weak coarsening of the augmented space for cohomology theories.

REMARK 6.1. Suppose that G is hyperbolic relative to \mathbb{P} . If $P = \emptyset$, then G is hyperbolic and thus Theorem 1.1 and Theorem 1.2 follow from Proposition 5.2 and Corollary 5.3. If $G \in \mathbb{P}$ then $\mathbb{P} = \{G\}$, thus Theorem 1.1 and Theorem 1.2 are trivial. It is well known that all elements are of infinite index of G if $G \notin \mathbb{P}$.

From now on, we assume that \mathbb{P} is not empty and all elements of \mathbb{P} are of infinite index in G .

6.1. The augmented space.

DEFINITION 6.2. Let (P, d) be a proper metric space. The *combinatorial horoball* based on P , denoted by $\mathcal{H}(P)$, is the graph defined as follows:

- (a) $\mathcal{H}(P)^{(0)} = P \times (\mathbb{N} \cup \{0\})$.
- (b) $\mathcal{H}(P)^{(1)}$ contains the following two type of edges:
 - (i) For each $l \in \mathbb{N} \cup \{0\}$ and $p, q \in P$, if $0 < d(p, q) \leq 2^l$ then there is a *horizontal edge* connecting (p, l) and (q, l) .
 - (ii) For each $l \in \mathbb{N} \cup \{0\}$ and $p \in P$, there is a *vertical edge* connecting (p, l) and $(p, l + 1)$.

We endow $\mathcal{H}(P)$ with the graph metric.

When P is a discrete proper metric space, $\mathcal{H}(P)$ is a proper geodesic space which is hyperbolic in the sense of Gromov. (See [15, Theorem 3.8]). It is easy to see that $\mathcal{H}(P)$ is coarsely flasque. The following is used in Section 7.

LEMMA 6.3. *Let P be a proper metric space. We suppose that P is discrete. Then the Gromov compactification of the combinatorial horoball $\mathcal{H}(P)$ is a one-point compactification of P . Thus the Gromov boundary of $\mathcal{H}(P)$ consists of one point, called the parabolic point of $\mathcal{H}(P)$.*

PROOF. See Lemma 3.11. in [15]. □

Let G be a finitely generated group with a finite family of infinite subgroups $\mathbb{P} = \{P_1, \dots, P_k\}$. We take a finite generating set \mathcal{S} for G . We assume that \mathcal{S} is symmetrized, so that $\mathcal{S} = \mathcal{S}^{-1}$. We endow G with the left-invariant word metric $d_{\mathcal{S}}$ with respect to \mathcal{S} .

DEFINITION 6.4. Let G and \mathbb{P} be as above. An order of the cosets of (G, \mathbb{P}) is a sequence $\{g_n\}_{n \in \mathbb{N}}$ such that $g_i = e$ for $i \in \{1, \dots, k\}$, and for each $r \in \{1, \dots, k\}$, the map $\mathbb{N} \rightarrow G/P_r : a \mapsto g_{ak+r}P_r$ is bijective. Thus the set of all cosets $\bigsqcup_{r=1}^k G/P_r$ is indexed by the map $\mathbb{N} \ni i \mapsto g_i P_{(i)}$. Here (i) denotes the remainder of i divided by k .

We fix an order $\{g_n\}_{n \in \mathbb{N}}$ of the cosets of (G, \mathbb{P}) . Each coset $g_i P_{(i)}$ has a proper metric d_i which is the restriction of $d_{\mathcal{S}}$. Let Γ be the Cayley graph of (G, \mathcal{S}) . There exists a natural embedding $\psi_i : \mathcal{H}(g_i P_{(i)}; \{0\}) \hookrightarrow \Gamma$ such that $\psi_i(x, 0) = x$ for all $x \in g_i P_{(i)}$.

DEFINITION 6.5. The augmented space $X(G, \mathbb{P}, \mathcal{S})$ is obtained by pasting $\mathcal{H}(g_i P_{(i)})$ to Γ by ψ_i for all $i \in \mathbb{N}$. Thus we can write it as follows:

$$X(G, \mathbb{P}, \mathcal{S}) := \Gamma \cup \bigcup_{i \in \mathbb{N}} \mathcal{H}(g_i P_{(i)}).$$

REMARK 6.6. The vertex set of $X(G, \mathbb{P}, \mathcal{S})$ can naturally identified with the disjoint union of G and the set of 3-tuple (i, p, l) , where $i \in \mathbb{N}$, $p \in g_i P_{(i)}$, and $l \in \mathbb{N}$. We sometimes denote $g \in g_i P_{(i)}$ by $(i, g, 0)$ for simplicity.

DEFINITION 6.7. A group G is hyperbolic relative to \mathbb{P} if the augmented space $X(G, \mathbb{P}, \mathcal{S})$ is hyperbolic in the sense of Gromov.

Groves and Manning [15] showed that the above definition is equivalent to the original one by Gromov.

6.2. Weak coarsening of relatively hyperbolic groups. In this section, we construct a topological counterpart of the augmented space, which is the key to the proof of Theorem 1.1 and Theorem 1.2. Let G be a finitely generated group which is hyperbolic relative to $\mathbb{P} = \{P_1, \dots, P_k\}$. Here we assume that for $r \in \{1, \dots, k\}$, each P_r admits a finite P_r -simplicial complex $\underline{E}P_r$ which is a universal space for proper actions. By [12, Appendix A], there exists a finite G -simplicial complex $\underline{E}G$ which is a universal space for proper actions such that all $\underline{E}P_r$ are embedded in $\underline{E}G$. We can assume that G is naturally embedded in the set of vertices of $\underline{E}G$ and $g_i P_{(i)}$ is embedded in $g_i \underline{E}P_{(i)}$.

We define an embedding $\eta_i : g_i \underline{E}P_{(i)} \times \{0\} \hookrightarrow \underline{E}G$ as $\eta_i(x, 0) = x$. We define a space $EX(G, \mathbb{P})$ in LCSH by pasting $g_i \underline{E}P_{(i)} \times [0, \infty)$ to $\underline{E}G$ by η_i for all $i \in \mathbb{N}$. Thus we can

write it as follows:

$$EX(G, \mathbb{P}) := \underline{EG} \cup \bigcup_{i \in \mathbb{N}} (g_i \underline{EP}_{(i)} \times [0, \infty)).$$

In the rest of this section, we show that $EX(G, \mathbb{P})$ is a weak coarsening of $X(G, \mathbb{P}, \mathcal{S})$, that is, $MX^*(X(G, \mathbb{P}, \mathcal{S})) \cong M^*(EX(G, \mathbb{P}))$. Here M^* is the K -theory K^* or the Alexander-Spanier cohomology with compact support H_c^* .

We can regard $EX(G, \mathbb{P})$ as a metric simplicial complex in the sense of [17, Definition 3.1]. However, the bounded coarse structure associated to this metric is not coarsely equivalent to $X(G, \mathbb{P}, \mathcal{S})$. Therefore we equip $EX(G, \mathbb{P})$ with a pull-back coarse structure as follows.

Let $X(G, \mathbb{P}, \mathcal{S})^{(0)}$ denote the 0-skeletons of $X(G, \mathbb{P}, \mathcal{S})$. Since G and P_r for $r = 1, \dots, k$ are embedded respectively into \underline{EG} and \underline{EP}_r , there is a natural embedding $\iota: X(G, \mathbb{P}, \mathcal{S})^{(0)} \hookrightarrow EX(G, \mathbb{P})$. We define a left inverse φ of ι as follows. We take a finite subcomplex $\Delta \subset \underline{EG}$ containing a fundamental domain of \underline{EG} . We may assume that $\Delta_r := \Delta \cap \underline{EP}_r$ contains a fundamental domain of \underline{EP}_r for $r = 1, \dots, k$ without loss of generality. Then we can write $EX(G, \mathbb{P})$ as follows.

$$EX(G, \mathbb{P}) = \bigcup_{g \in G} g\Delta \cup \bigcup_{i \in \mathbb{N}} \bigcup_{h \in P_{(i)}} g_i h \Delta_{(i)} \times (0, \infty).$$

For every $x \in \underline{EG}$, we choose $g_x \in G$ such that $x \in g_x \Delta$ and put $\varphi(x) := g_x \in \Gamma$. For $(x, t) \in g_i h \Delta_{(i)} \times (0, \infty)$, we put $\varphi(x, t) := (i, g_i h, [t]) \in \mathcal{H}(g_i P_{(i)})$ where $[t]$ denotes the integral part of t . We equip $EX(G, \mathbb{P})$ with a pullback coarse structure by φ . It is easy to see that ι and φ satisfy the conditions in Proposition 2.9 and Proposition 2.11. Therefore $EX(G, \mathbb{P})$ is a proper coarse space which is coarsely equivalent to $X(G, \mathbb{P}, \mathcal{S})$. By the construction, \underline{EG} and \underline{EP}_i with the restricted coarse structure are respectively coarsely equivalent to G and P_i . Since G is finitely generated, \underline{EG} has bounded geometry in the sense of [28, Definition 3.9] and is uniformly contractible in the sense of [28, Definition 5.24], and so does \underline{EP}_i .

In Section 2.3 and Section 3.1 of [12], the followings are defined.

- (a) An anti-Čech system $\{\mathcal{U}_n\}_n$ of $X(G, \mathbb{P}, \mathcal{S})$.
- (b) Coarsening maps $\alpha_n: \mathcal{U}_n \rightarrow \mathcal{U}_{n+1}$.
- (c) Subsets $\mathcal{X}_n, \mathcal{Y}_n, \mathcal{Z}_n$ of \mathcal{U}_n .
- (d) An anti-Čech system $\{E\mathcal{U}_n\}_n$ of $EX(G, \mathbb{P})$ in the sense of [28, Definition 5.36].
- (e) Simplicial maps $\phi_n: E\mathcal{U}_n \rightarrow \mathcal{U}_{n+1}$.

A partition of unity defines a continuous map $\psi: EX(G, \mathbb{P}) \rightarrow EU_1$. For $n \geq 3$, set $F_n := \alpha_{n-1} \circ \cdots \circ \alpha_2 \circ \phi_1 \circ \psi: EX(G, \mathbb{P}) \rightarrow |\mathcal{U}_n|$. We remark that the image of the restriction of F_n to \underline{EG} lies on $|\mathcal{X}_n|$. Then we have the following commutative diagram.

$$\begin{array}{ccccccc}
\longrightarrow & M^p(|\mathcal{U}_{n+1}|) & \longrightarrow & M^p(|\mathcal{X}_{n+1}|) \oplus M^p(|\mathcal{Y}_{n+1}|) & \longrightarrow & M^p(|\mathcal{Z}_{n+1}|) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & M^p(|\mathcal{U}_n|) & \longrightarrow & M^p(|\mathcal{X}_n|) \oplus M^p(|\mathcal{Y}_n|) & \longrightarrow & M^p(|\mathcal{Z}_n|) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & M^p(EX(G, \mathbb{P})) & \longrightarrow & M^p(\underline{EG}) & \longrightarrow & M^p(\bigsqcup_{i \in \mathbb{N}} g_i \underline{EP}_{(i)}) & \longrightarrow .
\end{array}$$

Here a map $M^p(|\mathcal{X}_n|) \oplus M^p(|\mathcal{Y}_n|) \rightarrow M^p(\underline{EG})$ is given by $(a, b) \mapsto F_n^*(a)$. Since \underline{EG} and \underline{EP}_i are of bounded geometry, uniformly contractible coarse spaces, by the same way as in the proof of [17, Proposition 3.8], taking subsequence if necessary, we can show that $\text{Im}[M^*(|\mathcal{X}_{n+1}|) \rightarrow M^*(|\mathcal{X}_n|)] \cong M^*(\underline{EG})$ and $\text{Im}[M^*(|\mathcal{Z}_{n+1}|) \rightarrow M^*(|\mathcal{Z}_n|)] \cong M^*(\bigsqcup_{i \in \mathbb{N}} g_i \underline{EP}_{(i)})$ for all $n \geq 1$. By the same argument as in the proof of [12, Lemma 2.7], we can show that $\text{Im}[M^*(|\mathcal{Y}_{n+1}|) \rightarrow M^*(|\mathcal{Y}_n|)] = 0$. Thus by diagram chasing, we have $\text{Im}[M^*(|\mathcal{U}_{n+1}|) \rightarrow M^*(|\mathcal{U}_n|)] \cong M^*(EX(G, \mathbb{P}))$ for all $n \geq 1$. Therefore we have $\varprojlim^1 M^*(|\mathcal{U}_n|) = 0$ and $\varprojlim M^*(|\mathcal{U}_n|) \cong M^*(EX(G, \mathbb{P}))$. By axiom (iv), we have the following conclusion.

PROPOSITION 6.8. *A space $EX(G, \mathbb{P})$ is a weak coarsening of $X(G, \mathbb{P}, \mathcal{S})$, that is, $MX^*(X(G, \mathbb{P}, \mathcal{S})) \cong M^*(EX(G, \mathbb{P}))$.*

We use the following notations introduced in [12]

$$\begin{aligned}
X_n &:= \Gamma \cup \bigcup_{i > n} \mathcal{H}(g_i P_{(i)}); \\
X_\infty &:= \bigcap_{n > 0} X_n; \\
EX_n &:= \underline{EG} \cup \bigcup_{i > n} (g_i \underline{EP}_{(i)} \times [0, \infty)); \\
EX_\infty &:= \bigcap_{n > 0} EX_n.
\end{aligned}$$

We remark that $X_0 = X(G, \mathbb{P}, \mathcal{S})$, $X_\infty = \Gamma$, $EX_0 = EX$ and $EX_\infty = \underline{EG}$. We note that the definition of X_n is slightly different from the one in [12], that is, the index is shifted by one. By the Mayer-Vietoris argument and Proposition 6.8, we have the following

PROPOSITION 6.9. *The following is commutative for all $n \in \mathbb{N}$*

$$\begin{array}{ccc} MX^*(X_n) & \xrightarrow{\cong} & M^*(EX_n) \\ \downarrow & & \downarrow \\ MX^*(X_{n+1}) & \xrightarrow{\cong} & M^*(EX_{n+1}). \end{array}$$

By the continuity of M^* , we have $\varinjlim M^*(EX_n) \cong M^*(\underline{EG})$. Since \underline{EG} is a finite model, we have $MX^*(G) \cong M^*(\underline{EG})$. Hence Proposition 6.9 implies the following.

COROLLARY 6.10. *We have an isomorphism $\varinjlim MX^*(X_n) \cong MX^*(G)$.*

6.3. Coarse assembly map and its dual. In this section, we give a proof of Theorem 1.1. The first statement is proved in [12]. The second statement is proved by a similar way. We suppose that \mathbb{P} satisfies the condition in Theorem 1.1, that is, the coarse co-assembly map is an isomorphism for all $P \in \mathbb{P}$.

By Proposition 5.2, the coarse co-assembly map $\mu^*: K_{*+1}(\mathfrak{c}^r(X_0)) \rightarrow KX^*(X_0)$ is an isomorphism. Since $X_n = X_{n+1} \cup \mathcal{H}(g_{n+1}P_{(n+1)})$ is an ω -excisive decomposition, by using the Mayer-Vietoris sequences, we can show that for all $n \in \mathbb{N}$, the coarse co-assembly map $\mu^*: K_{*+1}(\mathfrak{c}^r(X_n)) \rightarrow KX^*(X_n)$ is an isomorphism. Finally, by the continuity of the K -theory and Corollary 6.10 we have

$$\begin{array}{ccc} \varinjlim K_{*+1}(\mathfrak{c}^r(X_n)) & \xrightarrow{\cong} & \varinjlim KX^*(X_n) \\ \downarrow \cong & & \downarrow \cong \\ K_{*+1}(\mathfrak{c}^r(G)) & \longrightarrow & KX^*(G). \end{array}$$

The following is a somewhat converse statement of Theorem 1.1. However, we assume nothing on universal spaces for proper actions.

PROPOSITION 6.11. *Let G be a group which is hyperbolic relative to \mathbb{P} .*

- (a) *If $\mu_*: KX_*(G) \cong K_*(C^*(G))$, then $\mu_*: KX_*(P) \cong K_*(C^*(P))$ for every $P \in \mathbb{P}$.*
- (b) *If $\mu^*: K_{*-1}(\mathfrak{c}^r(G)) \cong KX^*(G)$, then $\mu^*: K_{*-1}(\mathfrak{c}^r(P)) \cong KX^*(P)$ for every $P \in \mathbb{P}$.*

PROOF. We fix $r \in \{1, \dots, k\}$. Set $A := \Gamma \cup \bigcup_{i \neq r} \mathcal{H}(g_i P_i)$ and $B := \Gamma \cup \mathcal{H}(g_r P_r)$. Then $X(G, \mathbb{P}, \mathcal{S}) = A \cup B$ and $B = \Gamma \cup \mathcal{H}(g_r P_r)$ are ω -excisive decompositions. By the Mayer-Vietoris arguments for $A \cup B$, we have $\mu_*: KX_*(B) \rightarrow K_*(C^*(B))$ and $\mu^*: K_*(\mathfrak{c}^r(B)) \rightarrow KX^*(B)$ are both isomorphisms. By the Mayer-Vietoris arguments for $B = \Gamma \cup \mathcal{H}(g_r P_r)$, we have $\mu_*: KX_*(\Gamma \cap \mathcal{H}(g_r P_r)) \rightarrow K_*(C^*(\Gamma \cap \mathcal{H}(g_r P_r)))$ and $\mu^*: K_*(\mathfrak{c}^r(\Gamma \cap \mathcal{H}(g_r P_r))) \rightarrow$

$KX^*(\Gamma \cap \mathcal{H}(g_r P_r))$ are both isomorphisms. Here we use the fact that $\mathcal{H}(g_r P_r)$ is coarsely flasque. Since $\Gamma \cap \mathcal{H}(g_r P_r)$ is coarsely equivalent to P_r , we have the conclusion. \square

7. CORONA OF RELATIVELY HYPERBOLIC GROUPS

In this section, we construct a corona of a relatively hyperbolic group. Here we sketch the construction. Let (G, \mathbb{P}) be a relatively hyperbolic group. We fix a generating set \mathcal{S} of G and an order $\{g_n\}_{n \in \mathbb{N}}$ of the cosets of (G, \mathbb{P}) in the sense of Definition 6.4. The Bowditch boundary $\partial X(G, \mathbb{P}, \mathcal{S})$ contains no information on a maximal parabolic subgroup P because all orbits by P go to a single parabolic point $s \in \partial X(G, \mathbb{P}, \mathcal{S})$. We remove the parabolic point s and equip $\partial X(G, \mathbb{P}, \mathcal{S}) \setminus \{s\}$ with a coarse structure which is coarsely equivalent to P . Let (W, ζ) be a corona of P . Then (W, ζ) is also a corona of $\partial X(G, \mathbb{P}, \mathcal{S}) \setminus \{s\}$. Thus we obtain a blown-up $\partial X(G, \mathbb{P}, \mathcal{S}) \setminus \{s\} \cup W$. Repeating this procedure to all parabolic points, we obtain a corona ∂X_∞ of G .

7.1. A coarse structure on the complement of a parabolic point. Let G be a group which is hyperbolic relative to \mathbb{P} . For $p, x, y \in X(G, \mathbb{P}, \mathcal{S})$, we denote by $(x|y)_p$ the Gromov product

$$(x|y)_p := \frac{1}{2}(d(x, p) + d(y, p) - d(x, y)).$$

We denote by $[x, y]$ a geodesic connecting x and y . Since $X(G, \mathbb{P}, \mathcal{S})$ is hyperbolic in the sense of Gromov, there exists $\delta_0 > 0$ such that every geodesic triangle is δ_0 -thin, that is, for any $x, y, z \in X(G, \mathbb{P}, \mathcal{S})$, and for any $u \in [x, y]$ and $v \in [x, z]$, if $d(x, u) = d(x, v) \leq (y|z)_x$, then $d(u, v) \leq \delta_0$. For details, see [13, Chapter 2].

Two geodesic rays in $X(G, \mathbb{P}, \mathcal{S})$ are said to be equivalent if the Hausdorff distance of their images is finite. For a geodesic ray $l: [0, \infty) \rightarrow X(G, \mathbb{P}, \mathcal{S})$, we denote by $[l]$ the equivalent class of l . We also write $l(\infty) = [l]$. The Gromov boundary of $X(G, \mathbb{P}, \mathcal{S})$, denoted by $\partial X(G, \mathbb{P}, \mathcal{S})$, consists of equivalent classes of geodesic rays. It carries a natural topology and $\overline{X}(G, \mathbb{P}, \mathcal{S}) := X(G, \mathbb{P}, \mathcal{S}) \cup \partial X(G, \mathbb{P}, \mathcal{S})$ is a compactification of $X(G, \mathbb{P}, \mathcal{S})$. The Gromov product is extended on $\overline{X}(G, \mathbb{P}, \mathcal{S})$ as follows. For $u, v \in \overline{X}(G, \mathbb{P}, \mathcal{S})$ and $p \in X(G, \mathbb{P}, \mathcal{S})$, we put

$$(u|v)_p := \sup \liminf_{i, j \rightarrow \infty} (x_i|y_j)_p$$

where the supremum is taken over all sequences $(x_i)_{i \geq 1}$ and $(y_i)_{i \geq 1}$ tending to u and v , respectively. For details, see [13, Chapter 7]. Let $l_0, l_1: [0, \infty) \rightarrow X(G, \mathbb{P}, \mathcal{S})$ be geodesic rays such that $p := l_0(0) = l_1(0)$. Then it is easy to see that $(l_0(s)|l_1(t))_p$ is non-decreasing

for all $s, t \geq 0$, thus we have $([l_0][l_1])_p \geq (l_0(s)|l_1(t))_p$ for all $s, t \geq 0$. The following is known.

LEMMA 7.1. *In the above setting, there exists t_0 such that for all $s, t \geq t_0$, we have $(l_0(s)|l_1(t))_p \geq ([l_0][l_1])_p - 3\delta_0$.*

PROOF. The lemma follows immediately from [13, Remark 7.2.8]. \square

The augmented space have the following *tautness*.

LEMMA 7.2. *The augmented space $X(G, \mathbb{P}, \mathcal{S})$ is taut, in fact, for any vertex $x \in X(G, \mathbb{P}, \mathcal{S})$, there exists a bi-infinite geodesic $l: (-\infty, \infty) \rightarrow X(G, \mathbb{P}, \mathcal{S})$ such that x lies on l .*

PROOF. Take any vertex $(i, g, n) \in X(G, \mathbb{P}, \mathcal{S})$. (See Remark 6.6, we often use this notation.) We choose $j \in \mathbb{N}$ such that $\mathcal{H}(g_i P_{(i)}) \cap \mathcal{H}(g_j P_{(j)}) = \emptyset$. Then we choose a shortest geodesic $\gamma: [0, a] \rightarrow X(G, \mathbb{P}, \mathcal{S})$ connecting $\mathcal{H}(g_i P_{(i)})$ and $\mathcal{H}(g_j P_{(j)})$. We remark that its end points $p := l(0)$ and $q := l(a)$ lie respectively on $g_i P_{(i)}$ and $g_j P_{(j)}$. We take the vertical ray $\gamma_-: [0, \infty) \rightarrow X(G, \mathbb{P}, \mathcal{S})$ from p to the parabolic point s_i of $\mathcal{H}(g_i P_{(i)})$. Also we take the vertical ray $\gamma_+: [0, \infty) \rightarrow X(G, \mathbb{P}, \mathcal{S})$ from q to parabolic point s_j of $\mathcal{H}(g_j P_{(j)})$. Then $\gamma_-([0, \infty)) \cup l([0, a]) \cup \gamma_+([0, \infty))$ is a bi-infinite geodesic from s_i to s_j . There exists $h \in G$ such that $(i, g, n) = (i, hp, n)$. Then (i, g, n) lies on the bi-infinite geodesic $h(\gamma_-([0, \infty)) \cup l([0, a]) \cup \gamma_+([0, \infty)))$. \square

Let N_{δ_0} be an integer greater than $\delta_0 + 1$. We fix $i \in \mathbb{N}$ and put $X^i := \Gamma \cup \bigcup_{j \neq i} \mathcal{H}(g_j P_{(j)})$. Set $e_i := (i, g_i, N_{\delta_0})$ as in remark 6.6. There exists a metric ρ_i on $\partial X(G, \mathbb{P}, \mathcal{S})$ which is compatible with the topology and satisfying that there exists constants $A, C > 0$ such that for any $u, v \in \partial X(G, \mathbb{P}, \mathcal{S})$, we have $A^{-1}e^{-C(u|v)e_i} \leq \rho_i(u, v) \leq Ae^{-C(u|v)e_i}$.

Let s_i be the parabolic point of the combinatorial horoball $\mathcal{H}(g_i P_{(i)})$. Set $\hat{P}_i := \partial X(G, \mathbb{P}, \mathcal{S}) \setminus \{s_i\}$. We equip \hat{P}_i with the subspace topology, as a subspace of $\partial X(G, \mathbb{P}, \mathcal{S})$. Let $l: \mathbb{R}_{\geq 0} \rightarrow X(G, \mathbb{P}, \mathcal{S})$ be a geodesic ray such that $l(0) = e_i$ and $l(\infty) \neq s_i$. We define $n_i(l) := \max\{n : l(n) \in g_i P_{(i)}\}$. By [15, Lemma 3.10], we can assume that geodesic segments $l([0, \infty)) \cap \mathcal{H}(g_i P_{(i)})$ consist of at most two vertical segments and a single horizontal segment of length at most 3.

LEMMA 7.3. *For any vertex $x \in X^i$, there exists a geodesic ray $l_x: [0, \infty) \rightarrow X(G, \mathbb{P}, \mathcal{S})$ and $t_x \in [0, \infty)$ such that $l_x(0) = e_i$, $l_x(\infty) \neq s_i$, $l_x(t_x) \in X^i$ and $d(x, l_x(t_x)) \leq 2\delta_0$.*

PROOF. By Lemma 7.2, there exists a geodesic $l: (-\infty, \infty) \rightarrow X(G, \mathbb{P}, \mathcal{S})$ and $s_x \in (-\infty, \infty)$ such that $x = l(s_x)$. Let $l_1, l_2: [0, \infty) \rightarrow X(G, \mathbb{P}, \mathcal{S})$ be geodesic rays such that $l_1(0) = l_2(0) = e_i$, $l_1(\infty) = l(-\infty)$, and $l_2(\infty) = l(\infty)$. We consider a geodesic triangle $l_1([0, \infty)) \cup l((-\infty, \infty)) \cup l_2([0, \infty))$. We can assume without loss of generality that $l(s_x)$ is contained in a δ_0 -neighborhood of $l_1([0, \infty))$. Therefore there exists $t'_x \in [0, \infty)$ such that $d(l(s_x), l_1(t'_x)) \leq \delta_0$. Suppose that $l_1(\infty) = s_i$. Then $l_1([0, \infty)) \subset \mathcal{H}(g_i P_{(i)}; [N_{\delta_0}, \infty))$, so x lies on the δ_0 -neighborhood of $\mathcal{H}(g_i P_{(i)}; [N_{\delta_0}, \infty))$. This contradicts that $N_{\delta_0} > \delta_0$. Thus $l_1(\infty) \neq s_i$. Set $l_x := l_1$. Then we have $d(x, l_x(t'_x)) \leq \delta_0$. If $l_x(t'_x) \in X^i$, then set $t_x := t'_x$, otherwise set $t_x := n_i(l_x)$. Then $d(x, l_x(t_x)) \leq 2\delta_0$. \square

In the rest of this section, we fix the following notations. For any vertex $x \in X^i$, we choose a geodesic ray l_x and $t_x \in [0, \infty)$ satisfying the statement of Lemma 7.3. For any point $u \in \hat{P}_i$, we choose a geodesic ray l^u such that $l^u(0) = e_i$ and $u = [l^u]$.

LEMMA 7.4. *Let $x \in X^i$ be a vertex. Set $u = [l_x]$. There exists $s_x \in [0, \infty)$ such that $l^u(s_x) \in X^i$ and $d(x, l^u(s_x)) \leq 4\delta_0$.*

PROOF. The Hausdorff distance of l_x and l^u is at most δ_0 . Thus there exists $s'_x \in [0, \infty)$ such that $d(l_x(t_x), l^u(s'_x)) \leq \delta_0$. If $l^u(s'_x) \in X^i$, we put $s_x = s'_x$, otherwise we put $s_x = n_i(l^u)$. Then by Lemma 7.3, $d(x, l^u(s_x)) \leq 4\delta_0$. \square

LEMMA 7.5. *Let $l_1: [0, a] \rightarrow X(G, \mathbb{P}, \mathcal{S})$ and $l_2: [0, b] \rightarrow X(G, \mathbb{P}, \mathcal{S})$ be geodesics such that $l_1(0) = l_2(0) = e_i$, and both of $l_1(a)$ and $l_1(b)$ lie on X^i . Then*

$$d(l_1(n_i(l_1)), l_2(n_i(l_2))) \leq d(l_1(a), l_2(b)) + 2\delta_0.$$

PROOF. Set $x := l_1(a)$, $y := l_2(b)$, $x' := l_1(n_i(l_1))$ and $y' := l_2(n_i(l_2))$. Here we remark that $x', y' \in g_i P_{(i)}$. Let r be an integer such that $d((i, x', r), (i, y', r)) \leq 1$. We choose g_{xy} such that $d((i, x', r), (i, g_{xy}, r)) = d((i, y', r), (i, g_{xy}, r)) = 1$. Set $p := (i, g_{xy}, r)$. We define $[p, x]$ as a geodesic consisting of a horizontal edge $\{(i, g_{xy}, r), (i, x', r)\}$, a vertical geodesic $[(i, x', r), (i, x', 0)]$ and $l_1([n_i(l_1), a])$. We also define a geodesic $[p, y]$ similarly. We consider a geodesic triangle $[p, x] \cup [x, y] \cup [p, y]$, which is δ_0 -thin. Here we remark that $d(p, x') = d(p, y') = r + 1$. If $r + 1 \leq (x|y)_p$, then

$$d(x', y') \leq \delta_0.$$

If $r + 1 > (x|y)_p$, then $d(x, x') \leq (y|p)_x$ since $(x|y)_p + (y|p)_x = d(p, x)$. Therefore, for a point $z \in [x, y]$ with $d(x, z) = d(x, x')$, we have $d(x', z) \leq \delta_0$. By the same reason, for a

point $w \in [x, y]$ with $d(y, w) = d(y, y')$, we have $d(y', w) \leq \delta_0$. Since $d(z, w) \leq d(x, y)$, we have

$$d(x', y') \leq d(x, y) + 2\delta_0.$$

□

We define a map $L_i: \hat{P}_i \rightarrow g_i P_{(i)}$ and $F_i: X^i \rightarrow \hat{P}_i$ as follows:

$$L_i(u) := l^u(n_i(l^u)) \text{ for } u \in \hat{P}_i;$$

$$F_i(x) := [l_x] \text{ for } x \in X^i.$$

LEMMA 7.6. *For any $x \in g_i P_{(i)}$, we have $d(x, L_i(F_i(x))) \leq 6\delta_0$.*

PROOF. Let $x \in g_i P_{(i)}$. Set $u = [l_x]$. By Lemma 7.4, there exists $s_x \in [0, \infty)$ such that $d(x, l^u(s_x)) \leq 4\delta_0$ and $l^u(s_x) \in X^i$. Then by Lemma 7.5, $d(x, L_i([l_x])) \leq d(x, l^u(s_x)) + 2\delta_0 \leq 6\delta_0$. □

LEMMA 7.7. *The composite $L_i \circ F_i$ is a large scale Lipschitz map, in fact, for any $x, y \in X^i$, we have*

$$d(L_i \circ F_i(x), L_i \circ F_i(y)) \leq d(x, y) + 10\delta_0.$$

PROOF. Let $x, y \in X^i$. Set $u = [l_x]$ and $v = [l_y]$. Then $L_i \circ F_i(x) = l^u(n_i(l^u))$ and $L_i \circ F_i(y) = l^v(n_i(l^v))$. By Lemma 7.4, there exist $s_x, s_y > 0$ such that $d(x, l^u(s_x)) \leq 4\delta_0$ and $d(y, l^v(s_y)) \leq 4\delta_0$. Then by Lemma 7.5,

$$d(L_i \circ F_i(x), L_i \circ F_i(y)) \leq d(x, y) + 10\delta_0.$$

□

We equip \hat{P}_i with the pullback coarse structure by L_i . We remark that $\hat{E} \subset \hat{P}_i \times \hat{P}_i$ is controlled if and only if there exists $R > 0$ such that for any $(u, v) \in \hat{E}$, we have $d(L_i(u), L_i(v)) < R$.

LEMMA 7.8. *Let $l: [0, \infty) \rightarrow X(G, \mathbb{P}, \mathcal{S})$ be a geodesic such that $l(0) = e_i$ and $l(\infty) \neq s_i$. Then for any $r > 0$, there exists t_r such that for all $t \geq t_r$, we have $d(l(t), \mathcal{H}(g_i P_{(i)})) > r$.*

PROOF. Suppose that there exists $r > 0$ such that $d(l(t), \mathcal{H}(g_i P_{(i)})) \leq r$ for all $t \geq 0$. Since the r -neighborhood of $\mathcal{H}(g_i P_{(i)})$ is coarsely equivalent to $\mathcal{H}(g_i P_{(i)})$, by Proposition 6.3, $l(t)$ converges to a parabolic point s_i as t goes to infinity. This contradicts the assumption. □

LEMMA 7.9. \hat{P}_i is a proper coarse space.

PROOF. We show that L_i satisfies the conditions in Proposition 2.11. Let $K \subset g_i P_{(i)}$ be a compact set. Fix $R > 0$ such that $K \subset B(e_i; R)$. Here $B(e_i; R)$ denotes a closed ball in $X(G, \mathbb{P}, \mathcal{S})$ of radius R centered at e_i . Let $u \in \hat{P}_i$. If $L_i(u) \in B(e_i; R)$, then $(u|s_i)_{e_i} \leq R$. Therefore we have

$$\begin{aligned} L_i^{-1}(K) &\subset \{u \in \hat{P}_i : d(e_i, L_i(u)) \leq R\} \\ &\subset \{u \in \hat{P}_i : \rho_i(s_i, u) \geq A^{-1}e^{-CR}\}. \end{aligned}$$

Thus $L_i^{-1}(K)$ is relatively compact.

We fix $u \in \hat{P}_i$. Since $u \neq s_i$, by Lemma 7.8, there exists $t_0 > 0$ such that for all $t > t_0$, we have $d(l^u(t), \mathcal{H}(g_i P_{(i)})) > 2\delta_0$. Let $v \in \hat{P}_i$ such that $(u|v)_{e_i} > t_0 + 3\delta_0$. By Lemma 7.1, there exists $s > 0$ such that $(l^u(s), l^v(s))_{e_i} \geq (u|v)_{e_i} - 3\delta_0$. Set $\tau = (l^u(s)|l^v(s))_{e_i}$. Since $\tau > t_0$, we have $d(l^u(\tau), \mathcal{H}(g_i P_{(i)})) > 2\delta_0$. Since a geodesic triangle

$$l^u([0, s]) \cup [l^u(s), l^v(s)] \cup l^v([0, s])$$

is δ_0 -thin, we have $d(l^u(\tau), l^v(\tau)) \leq \delta_0$. Thus, $d(l^v(\tau), \mathcal{H}(g_i P_{(i)})) > \delta_0$. Then we can apply Lemma 7.5 to l^u and l^v , so we have $d(L_i(u), L_i(v)) < 3\delta_0$. Thus, the inverse image $L_i^{-1}(B(L_i(u), 3\delta_0))$ contains a neighborhood $\{v \in \hat{P}_i : (u|v)_{e_i} > t_0 + 3\delta_0\}$ of u . Therefore L_i is pseudocontinuous. \square

PROPOSITION 7.10. \hat{P}_i and $g_i P_{(i)}$ are coarsely equivalent.

PROOF. We define a map $H_i : g_i P_{(i)} \rightarrow \hat{P}_i$ as the restriction of F_i , that is, $H_i(x) := [l_x]$ for $x \in g_i P_{(i)}$. Then by Lemma 7.6, the composite $L_i \circ H_i$ is close to the identity. So by Proposition 2.9, \hat{P}_i and $g_i P_{(i)}$ are coarsely equivalent. \square

PROPOSITION 7.11. For any Higson function $f \in C_h(\hat{P}_i)$, the pullback $F_i^* f := f \circ F_i$ is a Higson function on X^i .

PROOF. Let $f \in C_h(\hat{P}_i)$ be a Higson function. We fix $\epsilon > 0$ and $R > 0$. Let $\hat{E} := \{(u, v) : d(L_i(u), L_i(v)) < R + 10\delta_0\}$ be a controlled set of \hat{P}_i . There exists $S > 0$ such that for a bounded set $\hat{K} := \{u \in \hat{P}_i : d(e_i, L_i(u)) < S\}$ and for any $(u, v) \in \hat{E}$,

$$(7) \quad \text{if } (u, v) \notin \hat{K} \times \hat{K} \text{ then } |\mathbf{d}f(u, v)| < \epsilon.$$

On the other hand, since \hat{P}_i is a proper coarse space, \hat{K} is relatively compact. Thus the restriction $f|_{\hat{K}}$ is uniformly continuous on \hat{K} , so there exists $\theta > 0$ such that

$$(8) \quad \text{for any } u, v \in \hat{K}, \text{ if } \rho_i(u, v) < \theta \text{ then } |\mathbf{d}f(u, v)| < \epsilon.$$

Let $E_R := \{(x, y) : d(x, y) < R\}$ be a controlled set of X^i . By Lemma 7.7, we have $F_i(E_R) \subset \hat{E}$. Set

$$\begin{aligned} K' &:= \{x \in X^i : d(e_i, L_i \circ F_i(x)) < S\}; \\ T &:= -\frac{1}{C} \log\left(\frac{\theta}{A}\right) + R + 4\delta_0; \\ K &:= B(e_i, T). \end{aligned}$$

We remark that K' is unbounded. Let $(x, y) \in E_R$ such that $(x, y) \notin K \times K$. We first assume $(x, y) \notin K' \times K'$, then $(F_i(x), F_i(y)) \notin \hat{K} \times \hat{K}$. Thus by (7) we have $|\mathbf{d}F_i^* f(x, y)| = |\mathbf{d}f(F_i(x), F_i(y))| < \epsilon$. Next, we assume $(x, y) \in K' \times K'$. Since Lemma 7.3 implies

$$([l_x][l_y])_{e_i} \geq (l_x(t_x)|l_y(t_y))_{e_i} \geq T - R - 4\delta_0,$$

we have $\rho_i([l_x], [l_y]) < Ae^{-C(T-R-4\delta_0)} = \theta$. Then by (8) we have $|\mathbf{d}F_i^* f(x, y)| < \epsilon$. \square

By Proposition 7.11, F_i extends to a continuous map

$$hF_i: hX^i \rightarrow h\hat{P}_i.$$

Since the Gromov boundary is a corona, there exists a continuous map

$$\alpha: hX(G, \mathbb{P}, \mathcal{S}) \rightarrow \overline{X}(G, \mathbb{P}, \mathcal{S})$$

which is the identity on $X(G, \mathbb{P}, \mathcal{S})$. Since coarse embedding $X^i \hookrightarrow X(G, \mathbb{P}, \mathcal{S})$ induces an embedding $\nu X^i \hookrightarrow \nu X(G, \mathbb{P}, \mathcal{S})$, we regard νX^i as a subspace of $\nu X(G, \mathbb{P}, \mathcal{S})$. (See Proposition 2.15.)

LEMMA 7.12. *For any $y \in \nu X^i$, if $y \notin \alpha^{-1}(s_i)$ then we have $\alpha(y) = hF_i(y) \in \hat{P}_i$.*

PROOF. Let $y \in \nu X^i \setminus \alpha^{-1}(s_i)$. We choose a net $\{y_\lambda\}_{\lambda \in \Lambda}$ in X^i such that $y_\lambda \rightarrow y$. Then $\alpha(y_\lambda) \rightarrow \alpha(y)$. The restriction of α to $X(G, \mathbb{P}, \mathcal{S})$ is the identity, so

$$(F_i(y_\lambda)|\alpha(y))_{e_i} \geq (y_\lambda|\alpha(y))_{e_i} = (\alpha(y_\lambda)|\alpha(y))_{e_i} \rightarrow \infty.$$

Thus $F_i(y_\lambda) \rightarrow \alpha(y)$ in \hat{P}_i , so we have $hF_i(y) = \alpha(y)$. \square

7.2. Blow-up of parabolic points. In this section, we construct a corona of

$$X_n = \Gamma \cup \bigcup_{i>n} \mathcal{H}(g_i P_{(i)}).$$

For $r = 1, \dots, k$, let (W_r, ζ_r) be a corona of P_r . For $i \in \mathbb{N}$, set $W_i := W_{(i)}$ and $\zeta_i := \zeta_{(i)} \circ \nu g_i^{-1}$, where $\nu g_i^{-1}: \nu(g_i P_{(i)}) \rightarrow \nu P_{(i)}$ is an homeomorphism induced by an isometry $g_i P_{(i)} \ni x \mapsto g_i^{-1}x \in P_{(i)}$. Then (W_i, ζ_i) is a corona of $g_i P_{(i)}$. By Proposition 7.10, $\nu \hat{P}_i$ is homeomorphic to $\nu g_i P_{(i)}$, so we identify these two spaces. Thus we have a corona (W_i, ζ_i) of \hat{P}_i and a compact metrizable space $\hat{P}_i \cup W_i$. We recall that $\bar{\zeta}_i: h\hat{P}_i \rightarrow \hat{P}_i \cup W_i$ denotes an extension of ζ_i by the identity on \hat{P}_i . (See Section 2.2.)

We construct a corona of X_n by replacing s_i by W_i as follows. Set

$$(9) \quad \partial X_n(W_i; i = 1, \dots, n) := \partial X(G, \mathbb{P}, \mathcal{S}) \setminus \{s_1, \dots, s_n\} \sqcup \bigsqcup_{i=1}^n W_i.$$

We abbreviate $\partial X_n(W_i; i = 1, \dots, n)$ to ∂X_n . We equip ∂X_n with the weakest topology such that the maps $\sigma_i: \partial X_n \rightarrow \hat{P}_i \cup W_i$ are continuous for all $i \in \{1, \dots, n\}$. Here $\sigma_i(x) = s_j$ if $x \in W_j$ with $j \neq i$ and $\sigma_i(x) = x$ otherwise.

DEFINITION 7.13. The n -th blown-up of $\partial X(G, \mathbb{P}, \mathcal{S})$ with respect to $W_i, i = 1, \dots, n$ is a compact space $\partial X_n(W_i; i = 1, \dots, n)$ equipped with the above topology. The blown-up corona of $(G, \mathbb{P}, \{W_1, \dots, W_k\})$ is the projective limit $\partial X_\infty = \varprojlim \partial X_n$.

We also regard νX_n and νG as subspaces of $\nu X(G, \mathbb{P}, \mathcal{S})$. We define a map $\xi_n: \nu X_n \rightarrow \partial X_n$ as

$$\xi_n(x) := \begin{cases} \alpha(x) & \text{if } x \notin \bigcup_{i=1}^n \alpha^{-1}(s_i), \\ \bar{\zeta}_i \circ hF_i(x) & \text{if } x \in \alpha^{-1}(s_i) \text{ for } i = 1, \dots, n. \end{cases}$$

PROPOSITION 7.14. *The map $\xi_n: \nu X_n \rightarrow \partial X_n$ is continuous for all $n \in \mathbb{N} \cup \{\infty\}$. Thus ∂X_n and ∂X_∞ are respectively coronae of X_n and G . If ζ_i is surjective for $i = 1, \dots, k$, then so is ξ_n for all $n \in \mathbb{N} \cup \{\infty\}$.*

PROOF. It is enough to show that ξ_n is continuous on $\nu X_n \cap \alpha^{-1}(s_i)$. We fix $x \in \nu X_n \cap \alpha^{-1}(s_i)$. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net in νX_n such that $x_\lambda \rightarrow x$. If $x_\lambda \in \alpha^{-1}(s_i)$ then $\xi_n(x_\lambda) = \bar{\zeta}_i \circ hF_i(x_\lambda)$. If $x_\lambda \notin \alpha^{-1}(s_i)$ then by Lemma 7.12, $\xi_n(x_\lambda) = \alpha(x_\lambda) = \bar{\zeta}_i \circ hF_i(x_\lambda)$. Here we remark that $\bar{\zeta}_i$ is the identity on \hat{P}_i . Since $\bar{\zeta}_i \circ hF_i$ is continuous, we have $\xi(x_\lambda) \rightarrow \xi(x)$.

We suppose ζ_r is surjective for $r = 1, \dots, k$. We show that ξ_n is surjective for all $n \in \mathbb{N}$. In fact, we prove that the restriction $\xi_n: \nu G \rightarrow \partial X_n$ is surjective. Since the action of G on $\partial X_0 = \partial X(G, \mathbb{P}, \mathcal{S})$ is minimal ([2, Section 6]), $\xi_0: \nu G \rightarrow \partial X_0$ is surjective. We assume that $\xi_n: \nu G \rightarrow \partial X_n$ is surjective. Let $\pi_n: \partial X_{n+1} \rightarrow \partial X_n$ be a natural projection. Then we have $\xi_n = \pi_n \circ \xi_{n+1}$. Let $x \in \partial X_{n+1}$. If $x \in \pi_n^{-1}(s_{n+1}) = W_{n+1}$, then there exists y in $\nu(g_{n+1}P_{(n+1)})$ such that $\xi_{n+1}(y) = x$, where we regard $\nu(g_{n+1}P_{(n+1)})$ as a subspace of νG . Otherwise, there exists $y' \in \nu G$ such that $\pi_n(x) = \xi_n(y') = \pi_n(\xi_{n+1}(y'))$. Then we have $\xi_{n+1}(y') = x$ since the restriction of π_n to the complement of $\pi_n^{-1}(s_{n+1})$ is injective. \square

8. THE TRANSGRESSION MAPS

Let M^* be the K -theory K^* or the Alexander-Spanier cohomology with compact support H_c^* . Let G be a group which is hyperbolic relative to \mathbb{P} satisfying the condition of Theorem 1.2. Let $\{g_n\}_{n \in \mathbb{N}}$ be an order of the cosets of (G, \mathbb{P}) . Let s_i be the parabolic point of $g_i P_{(i)}$. Let X_n and EX_n be the one defined in Section 6.3. We can choose a map $\varphi_n: EX_n \rightarrow X_n$ such that the pullback coarse structure is proper and the φ_n is a coarse equivalence. (See Section 6.2.) Therefore we can regard a corona of X_n as that of EX_n .

For a compact space Z , we denote by $\mathcal{C}Z$ a closed cone of Z , that is, $\mathcal{C}Z = Z \times [0, 1] / \sim$ where $(z, 1) \sim (z', 1)$ for all z, z' in Z . Let (W_i, ζ_i) be a corona of $g_i P_{(i)}$ as in Section 7.2. Let $\partial X_n = \partial X_n(W_i; i = 1, \dots, n)$ be the n -th blown-up of $\partial X(G, \mathbb{P}, \mathcal{S})$. Let S_n be a space obtained by pasting $\mathcal{C}W_{n+1}$ on ∂X_{n+1} along W_{n+1} .

$$S_n := \partial X_{n+1} \cup \mathcal{C}W_{n+1}.$$

LEMMA 8.1. *The natural quotient map $S_n \rightarrow \partial X_n$ which sends $\mathcal{C}W_{n+1}$ to the parabolic point s_{n+1} induces an isomorphism $M^*(\partial X_n) \cong M^*(S_n)$.*

PROOF. The lemma follows from the strong excision property. (See [30, Chapter 6, Section 6] for the case of Alexander-Spanier cohomology.) \square

We use the following notations.

$$\begin{aligned} E\mathcal{H}(g_i P_{(i)}) &:= g_i E P_{(i)} \times [0, \infty); \\ \overline{E\mathcal{H}(g_i P_{(i)})} &:= \mathcal{C}(g_i \underline{E} P_{(i)} \cup_{\zeta_i} W_i). \end{aligned}$$

Then $\overline{E\mathcal{H}(g_i P_{(i)})}$ is a compactification of $E\mathcal{H}(g_i P_{(i)})$ and $\overline{E\mathcal{H}(g_i P_{(i)})} \setminus E\mathcal{H}(g_i P_{(i)}) = \mathcal{C}W_i$. We remark that $\overline{E\mathcal{H}(g_i P_{(i)})}$ is not any coarse compactification of $E\mathcal{H}(g_i P_{(i)})$.

PROPOSITION 8.2. *We suppose that the boundary map $\partial: \tilde{M}^{*-1}(W_i) \rightarrow M^*(\underline{E}P_i)$ is an isomorphism for $i = 1, \dots, k$. Then $\partial: \tilde{M}^{*-1}(\partial X_n) \rightarrow M^*(EX_n)$ is an isomorphism for all $n \geq 1$.*

PROOF. Since $X(G, \mathbb{P}, \mathcal{S})$ is hyperbolic and $\partial X(G, \mathbb{P}, \mathcal{S})$ is its Gromov boundary, by Corollary 5.3 and Lemma 8.1, the boundary map induces an isomorphism

$$\tilde{M}^{*-1}(S_0) \cong M^*(EX(G, \mathbb{P})).$$

The proposition inductively follows from Lemma 8.1 and Mayer-Vietoris sequences for $S_n = \partial X_{n+1} \cup \mathcal{C}W_{n+1}$ and for $EX_n = EX_{n+1} \cup E\mathcal{H}(g_{n+1}P_{(n+1)})$:

$$\begin{array}{ccccccc} \longrightarrow & \tilde{M}^{q-1}(S_n) & \longrightarrow & \tilde{M}^{q-1}(\partial X_{n+1}) \oplus \tilde{M}^{q-1}(\mathcal{C}W_{n+1}) & \longrightarrow & \tilde{M}^{q-1}(W_{n+1}) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & M^q(EX_n) & \longrightarrow & M^q(EX_{n+1}) \oplus M^q(E\mathcal{H}(g_{n+1}P_{(n+1)})) & \longrightarrow & M^q(g_{n+1}\underline{E}P_{(n+1)}) & \longrightarrow . \end{array}$$

Here we remark that $\tilde{M}^{q-1}(\mathcal{C}W_{n+1}) = M^q(E\mathcal{H}(g_{n+1}P_{(n+1)})) = 0$. \square

8.1. Proof of Theorem 1.2. Let M^* be the compactly supported Alexander-Spanier cohomology or the K -theory. Let (W_r, ζ_r) be a corona of P_r for $r = 1, \dots, k$. We remark that the boundary map $\partial: \tilde{M}^{*-1}(W_i) \rightarrow M^*(g_i \underline{E}P_i)$ is an isomorphism if and only if so is the transgression map $T_{W_i}: \tilde{M}^{*-1}(W_i) \rightarrow MX^*(g_i P_i)$. A similar statement for K -homology holds. By the continuity of M^* , we have $\tilde{M}^{*-1}(\partial X_\infty) \cong \varinjlim \tilde{M}^{*-1}(\partial X_n)$. Therefore, if $T_{W_r}: \tilde{M}^{*-1}(W_r) \rightarrow MX^*(P_r)$ is an isomorphism for all $r = 1, \dots, k$, then by Proposition 8.2 and Corollary 6.10, we have $\tilde{M}^{*-1}(\partial X_\infty) \cong MX^*(G)$.

If $T_{W_r}: KX_*(P_r) \rightarrow \tilde{K}_{*-1}(W_r)$ is an isomorphism for all $r = 1, \dots, k$, then, by the same way as in the proof of Proposition 8.2, we can show that $K_*(EX_n) \cong \tilde{K}_{*-1}(\partial X_n)$ for all $n \in \mathbb{N}$. By the Milnor exact sequence for $K_*(EX_n)$ and $\tilde{K}_{*-1}(\partial X_n)$, we have $KX_*(G) \cong \tilde{K}_{*-1}(\partial X_\infty)$.

9. APPLICATION

As an application of Theorems 1.1, 1.2 and Appendix A, we show the following.

COROLLARY 9.1. *Let G be a non-uniform lattice of a rank one symmetric space X of non-compact type. Suppose that the dimension of X is m . Then we have a corona ∂G of*

G and the following:

$$\begin{aligned} K_p(C^*(G)) &\cong KX_p(G) \cong \tilde{K}_{p-1}(\partial G) \cong \begin{cases} \prod_{i \in \mathbb{N}} \mathbb{Z} & (p = m-1) \\ 0 & (p = m) \end{cases}, \\ K^{p-1}(\mathfrak{c}^r(G)) &\cong KX^p(G) \cong \tilde{K}^{p-1}(\partial G) \cong \begin{cases} \bigoplus_{i \in \mathbb{N}} \mathbb{Z} & (p = m-1) \\ 0 & (p = m) \end{cases}, \\ HX^p(G) &\cong \tilde{H}^{p-1}(\partial G) \cong \begin{cases} \bigoplus_{i \in \mathbb{N}} \mathbb{Z} & (p = m-1) \\ 0 & (p \neq m-1) \end{cases}. \end{aligned}$$

REMARK 9.2. It is already known that the coarse assembly map and the coarse co-assembly map for G in the above are isomorphisms. Indeed G is known to be hyperbolic relative to a family of virtually nilpotent subgroups [11, Theorem 5.1] and thus we can use [33, Theorem 1.1] and [6, Section 1]. On the other hand when X is not any real hyperbolic space, that is, either a complex hyperbolic space, a quaternionic hyperbolic space or an octonionic plane, existence of a corona of G satisfying the above assertion is new in our knowledge. We note that each parabolic subgroup of G is virtually nilpotent but not virtually Abelian and thus G is neither CAT(0)-group nor hyperbolic group (see [24, Theorem A]). If X is a real hyperbolic space, then there is an alternative proof as follows. We take a truncated space X' which is a CAT(0)-space with a proper isometric cocompact action of G ([3, Corollary 11.28]) and consider its visual boundary $\partial_v X'$. Then $\partial_v X'$ satisfies the desired properties. Here we use the facts that X' is a ‘coarsening’ of G (see [17, Proposition 3.8 and Remark in p.233] and Section 3), $X' \cup \partial_v X'$ is contractible (see [31, Lemma 4.6.1]) and the visual boundary $\partial_v X'$ is an $(m-2)$ -dimensional Sierpinski curve ([29, Corollaries 4.2, 4.3] and its Mathematical review by Craig R. Guilbault). We omit the details.

9.1. Proof of Corollary 9.1. Let G be a lattice of a rank one symmetric space X of non-compact type. Suppose that the dimension of X is m . The space X is known to be either a real hyperbolic space, a complex hyperbolic space, a quaternionic hyperbolic space or an octonionic plane. Hence X is hyperbolic in the sense of Gromov and its Gromov boundary $\partial_G X$ is homeomorphic to S^{m-1} . If G is a uniform lattice of X , then

$\partial_G X$ is regarded as a corona of G and we have that

$$\begin{aligned} K_p(C^*(G)) &\cong KX_p(G) \cong \tilde{K}_{p-1}(\partial_G X) \cong \begin{cases} \mathbb{Z} & (p = m) \\ 0 & (p = m - 1) \end{cases}, \\ K^{p-1}(\mathfrak{c}^r(G)) &\cong KX^p(G) \cong \tilde{K}^{p-1}(\partial_G X) \cong \begin{cases} \mathbb{Z} & (p = m) \\ 0 & (p = m - 1) \end{cases}, \\ HX^p(G) &\cong \tilde{H}^{p-1}(\partial_G X) \cong \begin{cases} \mathbb{Z} & (p = m) \\ 0 & (p \neq m) \end{cases}. \end{aligned}$$

in view of Section 5.2.

Now we prove Corollary 9.1 which deals with the case where G is a non-uniform lattice of X .

Proof of Corollary 9.1. We take a set \mathbb{P} of representatives of conjugacy invariant classes of parabolic subgroups of G . Then \mathbb{P} is a finite family, and G is hyperbolic relative to \mathbb{P} ([11, Theorem 5.1]). The Gromov boundary $\partial_G X$ of X is the Bowditch boundary of (G, \mathbb{P}) . Moreover every element $P_r \in \mathbb{P}$ has a finite index subgroup P'_r which is a uniform lattice of an $(m-1)$ -dimensional Heisenberg-type (H-type for short) group (see [21, Proposition 1.1]). Since H-type groups are examples of Carnot groups and P_r is coarsely equivalent to P'_r , we have that \mathbb{P} satisfies the assumptions in Theorems 1.1 and 1.2 by Appendix A. Indeed we take a corona W_r of P'_r in Appendix A, which is homeomorphic to S^{m-2} and satisfies

$$\begin{aligned} K_*(C^*(P_r)) &\cong KX_*(P_r) \cong \tilde{K}_{*-1}(W_r), \\ K^{*-1}(\mathfrak{c}^r(P_r)) &\cong KX^*(P_r) \cong \tilde{K}^{*-1}(W_r), \\ HX^*(P_r) &\cong \tilde{H}^{*-1}(W_r). \end{aligned}$$

We define ∂G as the blown-up boundary of $(G, \mathbb{P}, \{W_r\})$. Then Theorems 1.1 and 1.2 imply the assertion except for concrete computations.

Now we describe ∂G . Take a finite generating set \mathcal{S} of G . Then we have a G -equivariant homeomorphism $\partial X(G, \mathbb{P}, \mathcal{S}) \cong \partial_G X$ by uniqueness of the Bowditch boundary of a relatively hyperbolic group (see [2, Section 9]). We note that \mathbb{P} is not empty because G is a non-uniform lattice of X . Then for every $n \in \mathbb{N}$, the n -th blown-up ∂X_n in Proof of Theorem 1.2 is homeomorphic to a complement of the interior of n disjoint closed balls in S^{m-1} . Since ∂X_∞ is the projective limit of ∂X_n , we can compute its reduced K -homology, reduced K -theory and reduced cohomology. \square

APPENDIX A. CORONAE AND COARSE COHOMOLOGIES OF LATTICES OF CARNOT GROUPS

Hyperbolic groups in the sense of Gromov and $\text{CAT}(0)$ groups are examples whose (generalized) coarse cohomologies and (generalized) coarse homologies are captured by their coronae (refer to [17], [31, Proposition 4.6.3] and Section 5.2). In our knowledge, there are no other classes of groups in literature. In this appendix, we consider another class of groups, indeed the class of uniform lattices of Carnot groups. Heisenberg groups (more generally H-type groups) with Carnot-Carathéodory metrics are typical examples of Carnot groups. See [20] about H-type groups. Also see for example [4] about Carnot groups.

We recall definitions and properties related to Carnot groups. Let N be an n -dimensional simply connected nilpotent Lie group and \mathfrak{n} be its Lie algebra. Then the following are well-known:

- $\exp : \mathfrak{n} \rightarrow N$ is an analytic homeomorphism (see for example [5, Theorem 1.2.1]);
- every lattice of N is uniform.

Suppose that \mathfrak{n} satisfies a stratification $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_l$ such that $\mathfrak{n}_l \neq \{0\}$, $[\mathfrak{n}_j, \mathfrak{n}_1] = \mathfrak{n}_{j+1}$ for any $j \in \{1, \dots, l-1\}$. Fix an inner product $(-, -)$ on \mathfrak{n}_1 . Then N has a distribution with a left invariant inner product by left translations of \mathfrak{n}_1 with $(-, -)$. A curve c on N is said to be horizontal if it is tangent to the distribution almost everywhere. The length of a horizontal curve $c : [t_1, t_2] \rightarrow N$ is defined as

$$L(c) := \int_{t_1}^{t_2} (c'(t), c'(t))^{\frac{1}{2}} dt.$$

Since \mathfrak{n}_1 generates \mathfrak{n} , it follows from Chow's theorem that any two points of N can be joined by a horizontal curve. For any $a_1, a_2 \in N$, we define $d_{cc}(a_1, a_2)$ as the infimum of length of horizontal curves from a_1 to a_2 . Then d_{cc} is a metric on N , which is called a Carnot-Carathéodory metric (or a sub-Riemannian metric). We call a pair (N, d_{cc}) a Carnot group. We define $\mathbb{R}_{>0} \times \mathfrak{n} \ni (t, v) \mapsto \bar{\delta}_t(v) \in \mathfrak{n}$ as $\bar{\delta}_t(v) := \sum_{j=1}^l t^j v_j$ where $v_j \in \mathfrak{n}_j$ with $v = \sum_{j=1}^l v_j$. Then we define an action $\mathbb{R}_{>0} \times N \ni (t, a) \mapsto \delta_t(a) \in N$ as $\delta_t(a) := \exp \bar{\delta}_t(v)$ where $v \in \mathfrak{n}$ with $a = \exp v$. The following are known facts which we freely use:

- d_{cc} is a left invariant proper metric on N ;
- $d_{cc}(\delta_t(a), \delta_t(a')) = td_{cc}(a, a')$ for any $a, a' \in N$ and $t \in \mathbb{R}_{>0}$;

- For each $t \in \mathbb{R}_{>0}$, $D(t) := \{a \in N \mid d_{cc}(o, a) \leq t\}$ is homeomorphic to a closed n -dimensional Euclidean ball.

First two statements are proved by definition. The final one follows from [22, Proposition 4.4.10].

For any $s, t \in \mathbb{R}_{>0}$ with $s < t$, we define a surjection $\pi_{s,t} : D(t) \rightarrow D(s)$ as $\pi_{s,t}(a) := a$ if $d_{cc}(o, a) \leq s$ and $\pi_{s,t}(a) := \delta_{s/d_{cc}(o,a)}(a)$ if $d_{cc}(o, a) > s$. Also for any $t \in \mathbb{R}_{>0}$, we define a surjection $\pi_t : N \rightarrow D(t)$ as $\pi_t(a) := a$ if $d_{cc}(o, a) \leq t$ and $\pi_t(a) := \delta_{t/d_{cc}(o,a)}(a)$ if $d_{cc}(o, a) > t$.

We consider a projective system consisting of $\{\pi_{s,t} : D(t) \rightarrow D(s)\}_{0 < s < t}$. Then the projective limit $\varprojlim D(t)$ contains N as an open dense subset by the map $\varprojlim \pi_t : N \rightarrow \varprojlim D(t)$. We put $\overline{N} := \varprojlim D(t)$ and $\partial_{cc}N := \overline{N} \setminus N$. Then \overline{N} is homeomorphic to an n -dimensional Euclidean closed ball and $\partial_{cc}N$ is homeomorphic to its boundary, that is, the $(n - 1)$ -dimensional sphere.

PROPOSITION A.1. *Let N and $\partial_{cc}N$ be as above. Let P be a uniform lattice of N . Then $\partial_{cc}N$ is regarded as a corona of P and we have the following:*

$$\begin{aligned} K_*(C^*(P)) &\cong KX_*(P) \cong \tilde{K}_{*-1}(\partial_{cc}N), \\ K^{*-1}(\mathfrak{c}^r(P)) &\cong KX^*(P) \cong \tilde{K}^{*-1}(\partial_{cc}N), \\ HX^*(P) &\cong \tilde{H}^{*-1}(\partial_{cc}N). \end{aligned}$$

PROOF. Note that N can be regarded as a universal space for (free) proper actions of P . Since P is amenable and thus has Yu's Property (A), so it is coarsely embeddable into a Hilbert space ([33, Theorem 2.2]). Hence the coarse assembly map and the coarse co-assembly map for P are isomorphisms ([33, Theorem 1.1] and [9, Theorem 9.2]).

We show that $\partial_{cc}N$ is a corona of N . We take $t \in \mathbb{R}_{>0}$ and $f \in C(D(t))$. Then it is enough to prove that $F := f \circ \pi_t$ is a Higson function on N . We take $\epsilon > 0$ and $R > 0$. Since $D(t + R)$ is compact, we have $\delta > 0$ such that $|F(a') - F(b')| < \epsilon$ for any $a', b' \in D(t + R)$ with $d_{cc}(a', b') < \delta$. We take any $a, b \in N$ with $d_{cc}(a, b) < R$ and

$$r := \min\{r_a := d_{cc}(o, a), r_b := d_{cc}(o, b)\} > S := \max\{n, nR/\delta\}.$$

We put $a' = \delta_{n/r}a$ and $b' = \delta_{n/r}b$. Then we have $n \leq d_{cc}(o, a') = nr_a/r$, $d_{cc}(o, b') = nr_b/r < n + R$ and $F(a') = F(a)$, $F(b') = F(b)$. Since

$$d_{cc}(a', b') = d_{cc}(\delta_{\frac{n}{r}}a, \delta_{\frac{n}{r}}b) = \frac{n}{r}d_{cc}(a, b) < \frac{nR}{r} \leq \delta,$$

we have $|F(a) - F(b)| < \epsilon$.

When we consider the metric $d_{cc,P}$ on P induced by d_{cc} on N , this is a proper left invariant metric on P . Since $(P, d_{cc,P})$ and (N, d_{cc}) are coarsely equivalent, $\partial_{cc}N$ is regarded as a corona of P .

Now we regard N as a universal space for proper actions of P . Since N is uniformly contractible and has bounded geometry, the coarsening map and the character maps

$$K_*(N) \rightarrow KX_*(N), KX^*(N) \rightarrow K^*(N), HX^*(N) \rightarrow H_c^*(N)$$

are isomorphisms. (See Section 3.2, [9, Theorem 4.8] and [27, (3.33) Proposition]). Also since $\overline{N} = N \cup \partial_{cc}N$ is contractible, we have

$$K_*(N) \cong \tilde{K}_{*-1}(\partial_{cc}N), \tilde{K}^{*-1}(\partial_{cc}N) \cong K^*(N), \tilde{H}^{*-1}(\partial_{cc}N) \cong H_c^*(N).$$

Hence we have

$$KX_*(N) \cong \tilde{K}_{*-1}(\partial_{cc}N), \tilde{K}^{*-1}(\partial_{cc}N) \cong KX^*(N), \tilde{H}^{*-1}(\partial_{cc}N) \cong HX^*(N).$$

Since the inclusion from P to N is a coarse equivalence which covers the identity on $\partial_{cc}N$, we have the assertion. \square

APPENDIX B. MILNOR EXACT SEQUENCES BY PHILLIPS

In this appendix, we state a Milnor exact sequence by Phillips and give a proof for reader's convenience.

Phillips [25] studied the representable K -theory of the projective limit of σ - C^* -algebras. See [25] for representable K -theory of σ - C^* -algebras. He stated the following (in fact an equivariant version of the following) in [25, Theorem 5.8 (5)], which is a Milnor exact sequence for a projective system of σ - C^* -algebras.

PROPOSITION B.1. *Let $\{\pi_k : A_{k+1} \rightarrow A_k\}_{k \in \mathbb{N}}$ be a projective system of σ - C^* -algebras. Then we have the following functorial exact sequence for each $p \in \mathbb{Z}$.*

$$0 \rightarrow \varprojlim^1 RK_{p+1}(A_k) \rightarrow RK_p(\varprojlim A_k) \rightarrow \varprojlim RK_p(A_k) \rightarrow 0.$$

Phillips gives a proof under the condition that every π_k is surjective [25, Theorem 3.2]. In order to prove, we refer to it and to [23].

PROOF. We define

$$T := \{(F_k) \in \prod_{k \in \mathbb{N}} C([k-1, k], A_k) \mid F_k(k) = \pi_k(F_{k+1}(k)) \text{ for any } k \in \mathbb{N}\},$$

$$B_{k+1} := \{(F_k, a_{k+1}) \in C([k-1, k], A_k) \oplus A_{k+1} \mid F_k(k) = \pi_k(a_{k+1})\},$$

$$g_1 : T \ni (F_k) \mapsto (F_{2m-1}, F_{2m}(2m-1)) \in \prod_{m \in \mathbb{N}} B_{2m},$$

$$g_2 : T \ni (F_k) \mapsto (F_1(0), (F_{2m}, F_{2m+1}(2m))) \in A_1 \oplus \prod_{m \in \mathbb{N}} B_{2m+1},$$

$$f_1 : \prod_{m \in \mathbb{N}} B_{2m} \ni (F_1, a_2, F_3, a_4, \dots) \mapsto (F_1(0), a_2, F_3(2), a_4, \dots) \in \prod_{k \in \mathbb{N}} A_k,$$

$$f_2 : A_1 \oplus \prod_{m \in \mathbb{N}} B_{2m+1} \ni (a_1, F_2, a_3, F_4, \dots) \mapsto (a_1, F_2(1), a_3, F_3(2), \dots) \in \prod_{k \in \mathbb{N}} A_k,$$

$$\iota : \varprojlim A_k \ni (a_k) \mapsto ([k-1, k] \ni t \mapsto a_k) \in T,$$

$$\pi : T \ni (F_k) \mapsto (F_k(k)) \in \varprojlim A_k.$$

We have a pullback diagram

$$\begin{array}{ccc} T & \xrightarrow{g_1} & \prod_{m \in \mathbb{N}} B_{2m} \\ \downarrow g_2 & & \downarrow f_1 \\ A_1 \oplus \prod_{m \in \mathbb{N}} B_{2m+1} & \xrightarrow{f_2} & \prod_{k \in \mathbb{N}} A_k. \end{array}$$

Hence we have a Mayer-Vietoris sequence. Since $\pi \circ \iota = id$ and also $\iota \circ \pi$ and id are homotopic, ι gives a homotopy equivalence between the above pullback diagram and the following commutative diagram

$$\begin{array}{ccc} \varprojlim A_k & \longrightarrow & \prod_{m \in \mathbb{N}} A_{2m} \\ \downarrow & & \downarrow \\ \prod_{m \in \mathbb{N}} A_{2m-1} & \longrightarrow & \prod_{k \in \mathbb{N}} A_k. \end{array}$$

Now we have the desired functorial Milnor exact sequence. □

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Tomohiro Fukaya

Mathematical institute, Tohoku University, Sendai 980-8578, Japan

E-mail address: tomo@math.tohoku.ac.jp

Shin-ichi Oguni

Department of Mathematics, Faculty of Science, Ehime University, 2-5 Bunkyo-cho, Matsuyama, Ehime, 790-8577 Japan

E-mail address: oguni@math.sci.ehime-u.ac.jp